

COMPACTNESS PROPERTIES FOR GEOMETRIC FOURTH ORDER ELLIPTIC EQUATIONS WITH APPLICATION TO THE Q -CURVATURE FLOW

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ABSTRACT. We prove the compactness of solutions to general fourth order elliptic equations which are L^1 -perturbations of the Q -curvature equation on compact Riemannian 4-manifolds. Consequently, we prove the global existence and convergence of the Q -curvature flow on a generic class of Riemannian 4-manifolds. As a by product, we give a positive answer to an open question by A. Malchiodi [12] on the existence of bounded Palais-Smale sequences for the Q -curvature problem when the Paneitz operator is positive with trivial kernel.

1. INTRODUCTION AND STATEMENT OF THE RESULTS

On a Riemannian 4-manifold (M, g_0) , the Q -curvature of the metric g_0 is defined by

$$Q_0 = -\frac{1}{6} (\Delta_0 S_0 + S_0^2 - 3|\text{Ric}_0|^2), \quad (1.1)$$

where Δ_0 , S_0 and Ric_0 denote respectively, the Laplace-Beltrami operator, the scalar curvature and the Ricci tensor associated to the metric g_0 . A conformal change of the metric g_0 produces a metric $g = e^{2u}g_0$ having Q -curvature

$$Q_g = e^{-4u}(P_0 u + Q_0), \quad (1.2)$$

where P_0 is the Paneitz operator defined by

$$P_0 u = \Delta_0^2 u + \text{div}_0 \left[\left(\frac{2}{3} S_0 g_0 - 2 \text{Ric}_0 \right) du \right], \quad (1.3)$$

where div_0 denotes the divergence operator with respect to g_0 . The Paneitz operator is conformally invariant in the sense that if $g = e^{2u}g_0$, then the Paneitz operator with respect to g is given by $P_g = e^{-4u}P_0$. Through this paper, we will always assume that P_0 has trivial kernel, that is, its kernel consists only of constant functions. Some times, we will also need to assume that P_0 is positive, which means that for all $u \in C^\infty(M)$ we have

$$\int_M P_0 u \cdot u \, dV_0 \geq 0,$$

2000 *Mathematics Subject Classification.* 53A30 , 53C21 , 35K25.

Key words and phrases. Geometric PDE's, Variational method, Q -curvature.

where dV_0 is the volume element with respect to g_0 . We note here that both hypothesis are conformally invariant. From the following Chern-Gauss-Bonnet formula :

$$\int_M Q_g dV_g + \frac{1}{4} \int_M |W_g|^2 dV_g = 8\pi^2 \chi(M),$$

where $\chi(M)$ denotes the Euler-characteristic of M , W_g is the Weyl tensor of g and dV_g is the volume element with respect to g , we see that the total Q -curvature

$$k_0 := \int_M Q_0 dV_0 = \int_M Q_g dV_g \quad (1.4)$$

is also conformally invariant since the Weyl tensor is pointwise conformally invariant. We note here that formula (1.4) is also a direct consequence of (1.2).

The Q -curvature and the Paneitz operator have received much attention in recent years because of their role in four-dimensional conformal geometry and in mathematical physics. Similarly to the uniformization problem of surfaces, one of the interesting problems in the geometry of 4-manifolds is to ask if there exists a metric g on M conformal to g_0 and having constant Q -curvature ? By using the relation (1.2), the problem is equivalent to find a function $u \in C^\infty(M)$ satisfying the following partial differential equation

$$P_0 u + Q_0 = k_0 e^{4u}. \quad (1.5)$$

Equation (1.5) has a variational structure, and its solutions are critical points of the following functional on the Sobolev space $H^2(M)$:

$$E(u) := \frac{1}{2} \int_M P_0 u \cdot u dV_0 + \int_M Q_0 u dV_0 - \frac{k_0}{4} \log \left(\int_M e^{4u} dV_0 \right), \quad (1.6)$$

that we will call the Q -curvature functional through this paper. The main difficulty in the study of this functional is that in general it is not coercive, and it can be unbounded from below and from above. This is due to the large values that the total Q -curvature k_0 may have, and to the possibility of negative eigenvalues of the operator P_0 .

S.A. Chang and P. Yang [7] first studied Equation (1.5) by minimizing the functional E . They constructed conformal metrics of constant Q -curvature when the Paneitz operator P_0 is positive with trivial kernel and the total Q -curvature satisfies $k_0 < 16\pi^2$ which is the total Q -curvature of the Euclidean sphere \mathbb{S}^4 . The key point in their proof is that by using Adams inequality (see section 2), if we suppose that P_0 is positive with trivial kernel and $k_0 < 16\pi^2$, then the functional E is bounded from below, coercive and its critical points can be found as global minima. Later, Z.Djadli and A.Malchiodi [8] solved equation (1.5) when P_0 is not necessarily positive under the condition that the kernel of P_0 is trivial and $k_0 \neq 16k\pi^2$, $k \in \mathbb{N}^*$. They constructed a critical point of E by a mini-max scheme based on a result by A. Malchiodi [12], and independently by O. Druet and F. Robert [9], on the compactness of solutions to fourth order elliptic equations. For more details on the Q -curvature problem we refer the reader to [1], [2], [4], [10] and the references therein.

In [12] A. Malchiodi proved, by assuming $k_0 \notin 16\pi^2 \mathbb{N}^*$, the compactness of any sequence $(u_n)_n$ satisfying a C^0 -perturbation of equation (1.5) of the form

$$P_0 u_n + Q_n = k_n e^{4u_n} \quad (1.7)$$

where $k_n = \int_M Q_n dV_0$, by assuming

$$Q_n \xrightarrow{n \rightarrow +\infty} Q_0 \text{ in } C^0(M).$$

This result does not apply to Palais-Smale sequences for the functional E since for such sequences one needs H^{-2} -perturbations of equation (1.5). An open question was kept in [12]: do there exist bounded Palais-Smale sequences for the functional E ? One of the main result of the present paper is to give a positive answer to this question when the Paneitz operator is positive with trivial kernel. We prove first a compactness result of solutions to L^1 -perturbations of equation (1.5), that is, we need only to assume

$$Q_n \xrightarrow{n \rightarrow +\infty} Q_0 \text{ in } L^1(M)$$

in equation (1.7) above. Then we apply this result to study the solution of the heat flow equation associated with (1.5) since such an equation is a L^1 -perturbation of (1.5), giving thus a convergent Palais-Smale sequence for the functional E . Our first main result reads :

Theorem 1.1. *Let (M, g_0) be a compact Riemannian 4-manifold whose Paneitz operator has trivial kernel. Let $f \in C^0(M)$, and let $(u_n, f_n)_n$ be a sequence in $H^2(M) \times L^1(M)$ satisfying*

$$P_0 u_n + f_n = k_n e^{4u_n} \quad (1.8)$$

$$\int_M e^{4u_n} dV_0 = 1 \quad (1.9)$$

and

$$f_n \xrightarrow{n \rightarrow +\infty} f \text{ in } L^1(M), \quad (1.10)$$

where $k_n = \int_M f_n dV_0$. Then one of the following alternatives holds :

- 1) either the sequence $(e^{|u_n|})_n$ is bounded in $L^p(M)$ for all $p \in [1, +\infty)$,
- 2) or for a subsequence, that we still denote by $(u_n)_n$ for simplicity, there exist a finite number of points $a_1, \dots, a_m \in M$ and integers $l_1, \dots, l_m \in \mathbb{N}^*$ such that

$$\sum_{j=1}^m l_j = \frac{1}{16\pi^2} \int_M f dV_0 \quad (1.11)$$

and

$$e^{4u_n} \xrightarrow{n \rightarrow +\infty} \frac{16\pi^2}{\int_M f dV_0} \sum_{j=1}^m l_j \delta_{a_j}, \quad (1.12)$$

in the sense of measures, where δ_a stands for the Dirac mass at the point $a \in M$.

A particular case of Theorem 1.1 is the following result when $\int_M f dV_0 \notin 16\pi^2\mathbb{N}^*$.

Corollary 1.1. *Let $(u_n, f_n)_n$ be a sequence in $H^2(M) \times L^1(M)$ as in Theorem 1.1. If we assume in addition*

$$\int_M f dV_0 \notin 16\pi^2\mathbb{N}^*, \quad (1.13)$$

then the sequence $(e^{|u_n|})_n$ is bounded in $L^p(M)$ for all $p \in [1, +\infty)$, that is,

$$\int_M e^{p|u_n|} dV_0 \leq C_p,$$

where C_p is a positive constant depending on p but not on n .

It is clear that Corollary 1.1 is a particular case of Theorem 1.1 since if condition (1.13) is satisfied, then the second alternative in Theorem 1.1 does not occur.

As a direct consequence of Corollary 1.1, we recover the result of A. Malchiodi [12] and O. Druet-F. Robert [9] stated above. More precisely, we have the following corollary on the compactness of solutions to the Q -curvature equation (1.5) :

Corollary 1.2. *Let (M, g_0) be a compact Riemannian 4-manifold whose Paneitz operator has a trivial kernel, and assume that the total Q -curvature k_0 satisfies $k_0 \notin 16\pi^2\mathbb{N}^*$. Then for any $k \in \mathbb{N}$, there exists a constant C_k depending only on k and (M, g_0) such that for any solution $u \in H^2(M)$ of the Q -curvature equation*

$$P_0 u + Q_0 = k_0 e^{4u}$$

with the normalization $\int_M e^{4u} dV_0 = 1$, we have

$$\|u\|_{C^k(M)} \leq C_k.$$

We note here that when $k_0 \neq 0$, the normalization condition $\int_M e^{4u_n} dV_0 = 1$ is automatically satisfied from the Q -curvature equation.

Remark 1.1. 1) *The conclusions of Theorem 1.1 and Corollary 1.1 remain valid if we consider equations of the form*

$$P_0 u_n + f_n = h_n e^{4u_n}$$

where we assume $h_n \in C^0(M)$ such that $C^{-1} \leq h_n \leq C$ for some positive constant C independent of n . This can be checked by following the same proofs with some necessary slight modifications.

2) *One can easily check that in Corollary 1.1 a subsequence of $(u_n)_n$ converges strongly in $W^{2,p}(M)$ for all $p \in [1, 2)$ to a function u_∞ satisfying the following equation*

$$P_0 u_\infty + f = k_0 e^{4u_\infty},$$

where $k_0 = \int_M f dV_0$.

3) As it can be seen from our proofs, Theorem 1.1 and Corollary 1.1 remain valid if we consider L^1 -perturbations of the mean field equation on compact Riemannian surfaces. Indeed, the same arguments work if we replace the Paneitz operator by the Laplacian. Some related results concerning the mean field equation are proved by J-B. Castéras in his thesis [6].

The main difficulty in the study of equations like (1.8) in Theorem 1.1 is the appearance of the so-called bubbling phenomena due to the concentration of the volume of the conformal metric $g_n = e^{2u_n} g_0$. We prove that if such phenomena occur then there must be some volume quantization. An important tool in the proof of such a result is an integral Harnack type inequality that we will prove in section 3.

As stated above, the second main result of this paper concerns the evolution problem associated with equation (1.5). More precisely, we will consider the evolution of a metric g on M under the flow:

$$\begin{cases} \partial_t g = -(Q_g - \overline{Q}_g) g \\ g(0) = e^{2u_0} g_0, \quad u_0 \in C^\infty(M) \end{cases} \quad (1.14)$$

where

$$\overline{Q}_g = \frac{1}{\text{Vol}_g(M)} \int_M Q_g dV_g = \frac{k_0}{\text{Vol}_g(M)}$$

is the average of Q_g .

Since equation (1.14) preserves the conformal structure of M , then $g(t) = e^{2u(t)} g_0$, where $u(t) \in C^\infty(M)$ with initial condition $u(0) = u_0 \in C^\infty(M)$. For simplicity, we have used the notation $u(t) := u(\cdot, t)$, $t \in I$, for any function defined on $M \times I$, where I is a subset of \mathbb{R} . Thus the flow (1.14) takes the form

$$\begin{cases} \partial_t u = -\frac{1}{2} e^{-4u} (P_0 u + Q_0) + \frac{1}{2} \frac{k_0}{\int_M e^{4u} dV_0}, \\ u(0) = u_0. \end{cases} \quad (1.15)$$

It is clear that the first equation in (1.15) is parabolic since P_0 is an elliptic operator. Then by classical methods it admits a solution $u \in C^\infty(M \times [0, T))$ where $T \leq +\infty$ denotes the maximal time of existence. By integrating the first equation in (1.15) over M with respect to the volume element of $g(t)$, we see that the volume of M with respect to $g(t)$ remains constant, that is,

$$\int_M e^{4u(t)} dV_0 = \int_M e^{4u_0} dV_0, \quad \forall t \in [0, T). \quad (1.16)$$

If we multiply the first equation in (1.15) by $\partial_t u$ and integrating with respect to $g(t)$, we see that the functional E is decreasing along the flow :

$$\frac{d}{dt} E(u(t)) = -2 \int_M e^{4u(t)} |\partial_t u(t)|^2 dV_0, \quad \forall t \in [0, T). \quad (1.17)$$

As far as we know, the evolution problem (1.15) has been studied only in the case where the total Q -curvature k_0 satisfies $k_0 \leq 16\pi^2$ and P_0 is positive with trivial kernel. Indeed, S.Brendle [3] was the first who studied the Q -curvature flow by considering a more general flow (with prescribed Q -curvature function) on Riemannian manifolds of even dimension. In dimension four, his result corresponds to assume that P_0 is positive with trivial kernel and $k_0 < 16\pi^2$, and then he proved that (1.15) has a solution which is defined for all time ($T = +\infty$) and converges to a smooth function u_∞ such that the metric $g_\infty = e^{2u_\infty}$ has constant Q -curvature. When $k_0 = 16\pi^2$, he proved in [5] the global existence and the convergence of Q -curvature flow on the sphere \mathbb{S}^4 . We also mention here the work of A. Malchiodi and M. Struwe [13] where they consider the Q -curvature flow on \mathbb{S}^4 with a prescribed Q -curvature function f . They proved the global existence of the flow and studied its asymptotic behaviour under some assumptions on the critical points of f .

Our second main result in this paper is to study the flow (1.15) on Riemannian 4-manifolds with total curvature k_0 satisfying $k_0 \notin 16\pi^2\mathbb{N}^*$. In particular, we are able to allow k_0 to take values beyond the critical threshold $16\pi^2$. Our result is as follows :

Theorem 1.2. *Let (M, g_0) be a compact Riemannian 4-manifold whose Paneitz operator P_0 is positive with trivial kernel. For $u_0 \in C^\infty(M)$, let $u \in C^\infty(M \times [0, T))$ the solution of problem (1.15) defined on a maximal interval $[0, T)$. If*

$$\inf_{t \in [0, T)} E(u(t)) > -\infty, \quad (1.18)$$

where E is defined by (1.6), then $T = +\infty$, that is, $u(t)$ is globally defined on $[0, +\infty)$. Moreover, if in addition the total Q -curvature k_0 satisfies $k_0 \notin 16\pi^2\mathbb{N}^*$, then $u(t)$ converges in $C^\infty(M)$ as $t \rightarrow +\infty$, to a function $u_\infty \in C^\infty(M)$ satisfying the Q -curvature equation

$$P_0 u_\infty + Q_0 = \frac{k_0}{\int_M e^{4u_\infty} dV_0} e^{4u_\infty}.$$

It is natural to ask if there exist initial data $u_0 \in C^\infty(M)$ for which the solution $u(t)$ of problem (1.15) with $u(0) = u_0$, satisfies condition (1.18) in Theorem 1.2 ? As we will see below, we can always find initial data for which (1.18) is satisfied, and others for which it is not the case. We note here that the condition (1.18) is automatically satisfied if $k_0 \leq 16\pi^2$ and P_0 positive with trivial kernel since in this case the functional E is bounded from below by using Adams inequality (see section 2). First we have :

Theorem 1.3. *Let (M, g_0) be a compact Riemannian 4-manifold whose Paneitz operator P_0 is positive with trivial kernel, and suppose that the total Q -curvature k_0 satisfies $k_0 \notin 16\pi^2\mathbb{N}^*$. Then there exists at least one function $u_0 \in C^\infty(M)$ such that the solution $u(t)$ of problem (1.15) with $u(0) = u_0$, satisfies condition (1.18) in Theorem 1.2, that is,*

$$\inf_{t \in [0, T)} E(u(t)) > -\infty.$$

Thus, according to Theorem 1.2, $u(t)$ is globally defined on $[0, +\infty)$ and converges in $C^\infty(M)$ as $t \rightarrow +\infty$, to a function $u_\infty \in C^\infty(M)$ satisfying the Q -curvature equation

$$P_0 u_\infty + Q_0 = \frac{k_0}{\int_M e^{4u_\infty} dV_0} e^{4u_\infty}.$$

It follows from Theorem 1.3, by taking any real sequence $(t_n)_n$ such that $t_n \xrightarrow{n \rightarrow +\infty} +\infty$, that $(u(t_n))_n$ is a convergent Palais-Smale sequence for the Q -curvature functional E . This gives a positive answer to the open question by A. Malchiodi [12] stated above. In particular, Theorem 1.3 is a direct method to solve equation (1.5).

The following theorem gives a class of functions $u_0 \in C^\infty(M)$ for which condition (1.18) in Theorem 1.2 is not satisfied. More precisely, it gives a class of initial data for which the corresponding flow $u(t)$ blows up in finite or infinite time :

Theorem 1.4. *Let (M, g_0) be a compact Riemannian 4-manifold whose Paneitz operator P_0 is positive with trivial kernel, and suppose that the total Q -curvature k_0 satisfies $k_0 \notin 16\pi^2\mathbb{N}^*$. Then there exists a constant $\lambda \in \mathbb{R}$ such that for any $u_0 \in C^\infty(M)$ satisfying $E(u_0) \leq \lambda$, the corresponding solution $u(t)$ of (1.15) satisfies $\lim_{t \rightarrow T} E(u(t)) = -\infty$.*

One of the principal difficulties in the study of the Q -curvature flow (1.15) comes from the absence of a maximum principle for elliptic operator of high order (greater than 4). Our analysis is based on the proof of some delicate integral estimates concerning the parabolic equation (1.15), combined with the compactness of the solutions of the corresponding elliptic equation proved in Theorem 1.1.

2. PRELIMINARIES AND BLOW-UP ANALYSIS

We introduce in this section some basic tools on elliptic operators on Riemannian manifolds, and we recall some known results on the blow-up of solutions to general Q -curvature type equations.

Let (M, g_0) a smooth compact 4-Riemannian manifold without boundary. For the simplicity of notations, the Riemannian distance between two points $x, y \in M$ is denoted as in the Euclidean space, by $|x - y|$. If $x \in M$ and $r > 0$, we denote by $B_r(x)$ the geodesic ball in M of center x and radius r . If $x \in \mathbb{R}^n$, we denote in the same manner by $B_r(x)$ the Euclidean ball in \mathbb{R}^n of center x and radius r . The volume element of g_0 is denoted by dV_0 , and the volume of any measurable set $A \subset M$ is denoted by $|A|$. The Green function G associated to the Paneitz P_0 is a symmetric function $G \in C^\infty(M \times M \setminus D)$, where $D = \{(x, x) : x \in M\}$ is the diagonal of M , giving the inversion formula for Paneitz operator. That is, if $F \in L^1(M)$ with $\overline{F} = 0$, then u is a solution of

$$P_0 u = F \tag{2.1}$$

if and only if

$$u(x) = \bar{u} + \int_M G(x, y) F(y) dV_0(y), \tag{2.2}$$

where we denote by $\bar{h} := \frac{1}{|M|} \int_M h dV_0$ the average of any function $h \in L^1(M)$. We have the following asymptotics for G

$$G(x, y) = -\frac{1}{8\pi^2} \log |x - y| + R(x, y), \tag{2.3}$$

where $R \in C^0(M \times M)$.

The following proposition concerning solutions of equation (2.1) is proved in A. Malchiodi [12] by using the asymptotics of the Green function (see Lemma 2.3 in [12].)

Proposition 2.1. *Let $(u_n, F_n) \in H^2(M) \times L^1(M)$ satisfying*

$$P_0 u_n = F_n$$

with $\|F_n\|_{L^1(M)} \leq K$ for some constant K independent of n . Then for any $x \in M$, for any $r > 0$ (small enough), for any $j = 1, 2, 3$, and $p \in [1, 4/j)$, we have

$$\int_{B_r(x)} |\nabla^j u_n|^p dV_0 \leq C r^{4-jp},$$

where C is a positive constant depending on K, M, p but not on n .

We need also the following proposition proved in [12] :

Proposition 2.2. *Let $(u_n, F_n) \in H^2(M) \times L^1(M)$ satisfying*

$$P_0 u_n = F_n$$

with $\|F_n\|_{L^1(M)} \leq K$ for some constant K independent of n . Then :

1) either

$$\int_M e^{q(u_n - \bar{u}_n)} dV_0 \leq C$$

for some $q > 4$ and some $C > 0$ (independent of n),

2) or there exists a point $x \in M$ such that for any $r > 0$, we have

$$\liminf_{n \rightarrow +\infty} \int_{B_r(x)} |F_n| dV_0 \geq 8\pi^2.$$

Remark 2.1. *Proposition 2.2 remains valid if one replace the metric g_0 on M by a family of metric $(g_n)_n$ depending on n which is uniformly bounded in $C^k(M)$ for any $k \in \mathbb{N}$. The same result holds also if we replace M by any bounded open ball of \mathbb{R}^4 and assuming all the functions with compact support in this ball. Indeed, a bounded open ball of \mathbb{R}^4 can always be embedded in a torus for example.*

Now we shall give some basic properties of solutions to equation (2.1) when F is as in Theorem 1.1. That is, we consider a sequence $(u_n, f_n)_n$ in $H^2(M) \times L^1(M)$ satisfying

$$P_0 u_n + f_n = k_n e^{4u_n}, \tag{2.4}$$

such that

$$\int_M e^{4u_n} dV_0 = 1, \tag{2.5}$$

and

$$f_n \xrightarrow[n \rightarrow +\infty]{} f \text{ in } L^1(M), \tag{2.6}$$

with $f \in C^0(M)$, and where $k_n = \int_M f_n dV_0$.

First we state the following proposition which can easily be deduced from Proposition 2.2 above by setting $F_n = k_n e^{4u_n} - f_n$.

Proposition 2.3. *Let $(u_n, f_n)_n \in H^2(M) \times L^1(M)$ satisfying (2.4)-(2.6). Then :*

1) *either for any $p \geq 1$ we have for some constant $C_p > 0$ independent of n ,*

$$\int_M e^{p|u_n|} dV_0 \leq C_p$$

2) *or there exists a point $x \in M$ such that for any $r > 0$, we have*

$$k_0 \liminf_{n \rightarrow +\infty} \int_{B_r(x)} e^{4u_n} dV_0 \geq 8\pi^2 + o_r(1),$$

where $k_0 = \int_M f dV_0$, and where $o_r(1) \rightarrow 0$ as $r \rightarrow 0$.

If $(u_n)_n$ and $(f_n)_n$ are as above, $(x_n)_n$ is a sequence of points in M and $(r_n)_n$ a sequence of positive numbers such that $r_n \rightarrow 0$, we set

$$\widehat{u}_n(z) = u_n(\exp_{x_n}(r_n z)) + \log r_n, \quad z \in B_{\frac{\delta}{r_n}}(0), \quad (2.7)$$

where $B_{\frac{\delta}{r_n}}(0) \subset \mathbb{R}^4$ is the Euclidean ball of center 0 and radius $\frac{\delta}{r_n}$, and where δ is the injectivity radius of M . We note here that the ball $B_{\frac{\delta}{r_n}}(0)$ approaches \mathbb{R}^4 when $n \rightarrow +\infty$. As we will see later, it is useful to introduce the following quantities. Let $T_n : B_{\frac{\delta}{r_n}}(0) \rightarrow M$ defined by $T_n(z) = \exp_{x_n}(r_n z)$, and define a metric g_n on $B_{\frac{\delta}{r_n}}(0)$ by

$$g_n = r_n^{-2} T_n^* g. \quad (2.8)$$

It is not difficult to see that $g_n \xrightarrow{n \rightarrow +\infty} g_{\mathbb{R}^4}$ in $C^k(B_R(0))$ for all $k \in \mathbb{N}$ and all $R > 0$, where $g_{\mathbb{R}^4}$ is the standard Euclidean metric of \mathbb{R}^4 . An easy computation shows that \widehat{u}_n satisfies the following PDE in \mathbb{R}^4

$$P_{g_n} \widehat{u}_n + r_n^4 \widehat{f}_n = k_n e^{4\widehat{u}_n}, \quad (2.9)$$

where P_{g_n} is the Paneitz operator of the metric g_n in \mathbb{R}^4 and

$$\widehat{f}_n(z) = f_n(\exp_{x_n}(r_n z)), \quad z \in B_{\frac{\delta}{r_n}}(0). \quad (2.10)$$

We shall also use the following function on \mathbb{R}^4 , known as a standard bubble,

$$\xi_{z_0}(z) = \log \left(\frac{2\lambda}{1 + \lambda^2 |z - z_0|^2} \right) - \frac{1}{4} \log \left(\frac{k_0}{6} \right), \quad z \in \mathbb{R}^4, \quad (2.11)$$

where z_0 is a fixed point in \mathbb{R}^4 , $\lambda > 0$ is a positive constant, and $k_0 = \int_M f dV_0$ that we assume satisfying $k_0 > 0$.

The following is a slightly different definition of blow-up with respect to that given in [12] since we are considering more general equations.

Definition 2.1. Let $(u_n, f_n)_n \in H^2(M) \times L^1(M)$ satisfying (2.4)-(2.5). Let $(x_n)_n$ a sequence of points in M and $(r_n)_n$ a sequence of positive numbers such that $r_n \rightarrow 0$. We say that the sequence $(x_n, r_n)_n$ is a blow-up for $(u_n)_n$ if for some $z_0 \in \mathbb{R}^4$, we have for any $R > 0$,

$$\widehat{u}_n \xrightarrow{n \rightarrow +\infty} \xi_{z_0} \text{ in } \mathcal{D}'(B_R(0)) \text{ and } e^{4\widehat{u}_n} \xrightarrow{n \rightarrow +\infty} e^{4\xi_{z_0}} \text{ in } L^1(B_R(0)), \quad (2.12)$$

where \widehat{u}_n and ξ_{z_0} are defined by (2.7) and (2.11), and where $\mathcal{D}'(B_R(0))$ denotes the space of distributions on $B_R(0)$.

By using the above definition and the fact that $\int_{\mathbb{R}^4} e^{4\xi_{z_0}} dz = \frac{16\pi^2}{k_0}$, one can easily prove the following :

Proposition 2.4. Let $(u_n, f_n)_n \in H^2(M) \times L^1(M)$ satisfying (2.4)-(2.5), and let $(x_n, r_n)_n$ a blow-up for $(u_n)_n$. Then for any positive sequence $(\beta_n)_n$ such that $\beta_n \rightarrow +\infty$, there exists a positive sequence $(b_n)_n$ with $b_n \leq \beta_n$ and $b_n \rightarrow +\infty$ such that

$$\lim_{n \rightarrow +\infty} \int_{B_{b_n r_n}(x_n)} e^{4u_n} dV_0 = \frac{16\pi^2}{k_0}.$$

Now we will prove a proposition giving the existence of blow-ups for solutions to equation (2.4).

Proposition 2.5. Let $(u_n, f_n)_n \in H^2(M) \times L^1(M)$ satisfying (2.4)-(2.5). Suppose that there exist a sequence $(x_n)_n$ in M , sequences $(r_n)_n, (\widehat{r}_n)_n$ of positive numbers and a constant $\rho \in (0, \frac{\pi^2}{k_0}]$ (independent of n) such that

$$\widehat{r}_n \xrightarrow{n \rightarrow +\infty} 0, \quad \frac{r_n}{\widehat{r}_n} \xrightarrow{n \rightarrow +\infty} 0 \quad (2.13)$$

and for n large enough,

$$\int_{B_{r_n}(x_n)} e^{4u_n} dV_0 = \rho, \quad \int_{B_{r_n}(y)} e^{4u_n} dV_0 \leq \frac{\pi^2}{k_0} \quad \forall y \in B_{\widehat{r}_n}(x_n). \quad (2.14)$$

Then by passing to a subsequence, $(x_n, r_n)_n$ is a blow-up for $(u_n)_n$.

Proof. The proof follows closely that of Proposition 3.4 in A. Malchiodi [12]. But since we are considering more general equations, we shall give the detailed proof. It suffices to prove that for any fixed $R \geq 1$, a subsequence of $(\widehat{u}_n)_n$ converges in $W^{2,p}(B_R(0))$ to ξ_{z_0} for some $z_0 \in \mathbb{R}^4$ and some $p \geq 1$, and that $(e^{4\widehat{u}_n})_n$ converges to $e^{4\xi_{z_0}}$ in $L^1(B_R(0))$.

Let $R \geq 1$ be fixed large enough. In what follows, C denotes a positive constant depending on M and R but independent of n , whose values may change from line to line. Set

$$a_n = \frac{1}{|B_{2R}(0)|_{g_n}} \int_{B_{2R}(0)} \widehat{u}_n dV_n,$$

where dV_n is the volume element of the metric g_n defined in (2.8) above, and $|B_{2R}(0)|_{g_n}$ is the volume of the ball $B_{2R}(0)$ with respect to the metric g_n . Let now $\varphi \in C_0^\infty(B_{2R}(0))$ such that $\varphi = 1$

on $B_R(0)$, and define $v_n := \varphi(\widehat{u}_n - a_n)$. Then by using Proposition 2.1 and the Poincaré inequality, we have for any $p \in [1, 4/3)$ and $j = 1, 2, 3$,

$$\int_{B_{2R}(0)} |\nabla^j \widehat{u}_n|^p dV_n \leq C \quad \text{and} \quad \int_{B_{2R}(0)} |\widehat{u}_n - a_n|^p dV_n \leq C,$$

which implies that $(v_n)_n$ is bounded in $W^{3,p}(B_{2R}(0))$ for any $p \in [1, 4/3)$, that is

$$\|v_n\|_{W^{3,p}(B_{2R}(0))} \leq C. \quad (2.15)$$

(we recall here that the metric g_n is uniformly bounded in $C^k(B_R(0))$ for any $k \in \mathbb{N}$ and $R > 0$.)

Moreover we have

$$P_{g_n} v_n = -r_n^4 \varphi \widehat{f}_n + k_n \varphi e^{4\widehat{u}_n} + h_n, \quad (2.16)$$

where $h_n = L_n(\widehat{u}_n - a_n)$ for some third order linear operator L_n with uniformly bounded smooth coefficients, and where \widehat{f}_n is defined by (2.10). So, h_n is bounded in $L^p(B_{2R}(0))$ for any $p \in [1, 4/3)$. If we set $F_n = -r_n^4 \varphi \widehat{f}_n + k_n \varphi e^{4\widehat{u}_n} + h_n$, then by using (2.6), (2.13)-(2.14) and the fact that $(h_n)_n$ is bounded in $L^p(B_{2R}(0))$, one can check that there exists a constant $\delta_0 > 0$ independent of n such that for any $x \in B_{2R}(0)$ we have

$$\int_{B_{\delta_0}(x)} |F_n| dV_n \leq 2\pi^2 + o_n(1), \quad (2.17)$$

where $o_n(1) \rightarrow 0$ as $n \rightarrow +\infty$. Thus by applying Proposition 2.2 (with Remark 2.1) to equation (2.16), we obtain that

$$\int_{B_{2R}(0)} e^{q(v_n - \bar{v}_n)} dV_n \leq C \quad (2.18)$$

for some constant $q > 4$ independent of n , where $\bar{v}_n = \frac{1}{|B_{2R}(0)|_{g_n}} \int_{B_{2R}(0)} v_n dV_n$ is the average of v_n on $B_{2R}(0)$. Since $|\bar{v}_n| \leq C$ by (2.15), then (2.18) becomes

$$\int_{B_{2R}(0)} e^{qv_n} dV_n \leq C,$$

which gives

$$\int_{B_R(0)} e^{q(\widehat{u}_n - a_n)} dV_n \leq C, \quad (2.19)$$

since $v_n = \widehat{u}_n - a_n$ on $B_R(0)$.

By using Jensen inequality and the fact

$$\int_{B_{2R}(0)} e^{4\widehat{u}_n} dV_n = \int_{B_{2r_n R}(x_n)} e^{4u_n} dV_0 \leq \int_M e^{4u_n} dV_0 = 1, \quad (2.20)$$

we have

$$a_n \leq C. \quad (2.21)$$

Thus it follows from (2.19) and (2.21) that

$$\int_{B_R(0)} e^{q\widehat{u}_n} dV_n \leq C,$$

which gives

$$\int_{B_R(0)} e^{q\hat{u}_n} dx \leq C \quad (2.22)$$

since g_n is bounded in $C^k(B_R(0)) \forall k \in \mathbb{N}$, where dx is the Euclidean volume element of \mathbb{R}^4 .

Now, we have by (2.14) since $R \geq 1$

$$\rho = \int_{B_{r_n}(x_n)} e^{4u_n} dV_0 = \int_{B_1(0)} e^{4\hat{u}_n} dV_n \leq \int_{B_R(0)} e^{4\hat{u}_n} dV_n = e^{a_n} \int_{B_R(0)} e^{4(\hat{u}_n - a_n)} dV_n,$$

and since by (2.19) and Hölder's inequality we have $\int_{B_R(0)} e^{4(\hat{u}_n - a_n)} dV_n \leq C$ (recall here that g_n is bounded in $C^k(B_R(0)) \forall k \in \mathbb{N}$), then we obtain

$$e^{a_n} \geq C^{-1} \rho,$$

which together with (2.21) give

$$|a_n| \leq C. \quad (2.23)$$

It follows from (2.15) and (2.23), that $(\hat{u}_n)_n$ is bounded in $W^{3,p}(B_R(0))$ (for all $p \in [1, 4/3]$). Then by using Rellich-Kondrachov Theorem and passing to a subsequence, we have that $(\hat{u}_n)_n$ converges strongly in $W^{2,\alpha}(B_R(0))$ for all $\alpha \in [1, 2)$, and in $L^\beta(B_R(0))$ for all $\beta \in [1, +\infty)$, to a function $\hat{u}_\infty \in W^{3,p}(B_R(0))$ for all $p \in [1, 3/4]$. That is,

$$\forall \alpha \in [1, 2), \quad \|\hat{u}_n - \hat{u}_\infty\|_{W^{2,\alpha}(B_R(0))} \xrightarrow{n \rightarrow +\infty} 0 \quad (2.24)$$

and

$$\forall \beta \in [1, +\infty), \quad \|\hat{u}_n - \hat{u}_\infty\|_{L^\beta(B_R(0))} \xrightarrow{n \rightarrow +\infty} 0. \quad (2.25)$$

Thus it follows from (2.22) and (2.25) by using Hölder inequality (recall that $q > 4$ in (2.22)) that

$$\int_{B_R(0)} \left| e^{4\hat{u}_n} - e^{4\hat{u}_\infty} \right| dx \xrightarrow{n \rightarrow +\infty} 0. \quad (2.26)$$

Since \hat{u}_n satisfies equation (2.9), then by passing to the limit in this equation (in the distributional sense) where we use (2.26), we obtain

$$\Delta^2 \hat{u}_\infty = k_0 e^{4\hat{u}_\infty} \quad \text{in } \mathbb{R}^4, \quad (2.27)$$

where Δ is the Laplacian in \mathbb{R}^4 with respect to the Euclidean metric. By using (2.20), we see that \hat{u}_∞ satisfies also

$$\int_{B_R(0)} e^{4\hat{u}_\infty(z)} dz \leq 1,$$

and since $R \geq 1$ is arbitrary, then

$$\int_{\mathbb{R}^4} e^{4\hat{u}_\infty(z)} dz \leq 1. \quad (2.28)$$

The solutions of equation (2.27) satisfying (2.28) are classified in [11]. More precisely, it is proved in [11] that either

$$\hat{u}_\infty(z) = \log \left(\frac{2\lambda}{1 + \lambda^2 |z - z_0|^2} \right) - \frac{1}{4} \log \left(\frac{k_0}{6} \right) \quad (2.29)$$

for some $z_0 \in \mathbb{R}^4$ and $\lambda > 0$, or one has

$$-\Delta \widehat{u}_\infty(z) \xrightarrow{|z| \rightarrow +\infty} a \quad (2.30)$$

for some $a > 0$. But by using Proposition 2.1 and (2.24) (by taking $\alpha = 1$), one can easily check that for any $R \geq 1$,

$$\int_{B_R(0)} |\Delta \widehat{u}_\infty(z)| dz \leq CR^2. \quad (2.31)$$

On the other hand, if (2.30) occurs, then one has for R large enough

$$\int_{B_R(0)} |\Delta \widehat{u}_\infty(z)| dz \geq CaR^4$$

which contradicts (2.31). This proves that \widehat{u}_∞ is of the form (2.29). The proof of Proposition 2.4 is then complete. \square

We close this section with the following well known Adam's inequality (see [7]) :

Proposition 2.6. *Let (M, g_0) be a compact Riemannian 4-manifold whose Paneitz operator P_0 is positive with trivial kernel. Then for any $u \in H^2(M)$, we have*

$$\int_M e^{4(u-\overline{u})} dV_0 \leq C \exp \left(\frac{1}{4\pi^2} \int_M P_0 u \cdot u dV_0 \right), \quad (2.32)$$

where C is a positive constant depending only on (M, g_0) and where $\overline{u} = \frac{1}{|M|} \int_M u dV_0$ is the average of u on M .

3. INTEGRAL HARNACK TYPE INEQUALITY

In this section we shall prove an integral Harnack type inequality, which is an important tool in the proof of our results. In what follows, we set $f^+ = \max(f, 0)$ and $f^- = \max(-f, 0)$ for any function f on M . As already noticed in Section 2, in order to simplify the notations, we are denoting by $|x - y|$ the Riemannian distance between two points $x, y \in M$. The diameter of M is denoted by $\text{diam}(M)$.

Proposition 3.1. *Let $h \in C^0(M)$, and let $(u_n, h_n)_n$ be a sequence in $H^2(M) \times L^1(M)$ satisfying*

$$P_0 u_n + h = h_n, \quad (3.1)$$

where we suppose that

$$\lim_{n \rightarrow +\infty} \int_M h_n^- dV_0 = 0. \quad (3.2)$$

Let $(x_n, y_n)_n$ a sequence in $M \times M$, and let $(R_n)_n$ with $0 < R_n \leq \text{diam}(M)$ a sequence of positive numbers satisfying, for some constant C_0 independent of n ,

$$|x_n - y_n| \leq C_0 R_n \quad \text{and} \quad \int_{B_{2R_n}(y_n)} h_n^+ dV_0 \leq \pi^2. \quad (3.3)$$

Then for any sequence $(r_n)_n$ such that $0 < r_n \leq R_n$ we have

$$\int_{B_{R_n}(y_n)} e^{4u_n} dV_0 \leq C \left(\frac{r_n}{R_n} \right)^{-4 + \frac{1}{2\pi^2} \|h_n^+\|_{L^1(B_{r_n}(x_n))} + o_n(1)} \int_{B_{r_n}(x_n)} e^{4u_n} dV_0, \quad (3.4)$$

where $o_n(1) \rightarrow 0$ as $n \rightarrow +\infty$, and where C is a positive constant independent of n .

Remark 3.1. As it can be seen in the proof, one can replace π^2 in (3.3) in the above proposition by any positive constant $\rho < 4\pi^2$.

Proof. In what follows, C is a positive constant independent of n whose values may change from line to line. Also, to simplify the notations, we set $R = R_n$ and $r = r_n$. First, let us recall the asymptotic formula for the Green function (see section 2)

$$G(x, y) = -\frac{1}{8\pi^2} \log |x - y| + O(1), \quad \text{for all } x \neq y \text{ in } M. \quad (3.5)$$

From the Green representation formula (see section 2) we have, for any $x \in M$,

$$\begin{aligned} u_n(x) - \bar{u}_n &= \int_M G(x, y) h_n(y) dV_0(y) - \int_M G(x, y) h(y) dV_0(y) \\ &= \int_M G(x, y) h_n^+(y) dV_0(y) - \int_M G(x, y) h_n^-(y) dV_0(y) - \int_M G(x, y) h(y) dV_0(y). \end{aligned} \quad (3.6)$$

Since $h \in C^0(M)$, then by using (3.5) we have

$$\left| \int_M G(x, y) h(y) dV_0(y) \right| \leq C \|h\|_{L^\infty(M)}. \quad (3.7)$$

Thus it follows from (3.6) and (3.7) that for any $x \in M$,

$$\begin{aligned} u_n(x) - \bar{u}_n &\leq C \|h\|_{L^\infty(M)} + \int_{B_{2R}(y_n)} G(x, y) h_n^+(y) dV_0(y) + \int_{M \setminus B_{2R}(y_n)} G(x, y) h_n^+(y) dV_0(y) \\ &\quad - \int_M G(x, y) h_n^-(y) dV_0(y) \end{aligned}$$

which implies by integrating the function $e^{4(u_n(x) - \bar{u}_n)}$ on $B_R(y_n)$,

$$\begin{aligned} \int_{B_R(y_n)} e^{4(u_n - \bar{u}_n)} dV_0 &\leq e^{C \|h\|_{L^\infty(M)}} \int_{B_R(y_n)} \exp \left(\int_{B_{2R}(y_n)} 4G(x, y) h_n^+(y) dV_0(y) \right) dV_0(x) \\ &\times \exp \left(\sup_{x \in B_R(y_n)} \int_{M \setminus B_{2R}(y_n)} 4G(x, y) h_n^+(y) dV_0(y) - \inf_{z \in B_R(y_n)} \int_M 4G(z, y) h_n^-(y) dV_0(y) \right). \end{aligned} \quad (3.8)$$

By Jensen inequality, we have

$$\exp \left(\int_{B_{2R}(y_n)} 4G(x, y) h_n^+(y) dV_0(y) \right)$$

$$\leq \frac{1}{\|h_n^+\|_{L^1(B_{2R}(y_n))}} \int_{B_{2R}(y_n)} h_n^+(y) \exp\left(4G(x, y)\|h_n^+\|_{L^1(B_{2R}(y_n))}\right) dV_0(y),$$

and integrating this inequality on $B_R(y_n)$ (in the x -variable) and using Fubini's Theorem, we obtain

$$\begin{aligned} & \int_{B_R(y_n)} \exp\left(\int_{B_{2R}(y_n)} 4G(x, y)h_n^+(y)dV_0(y)\right) dV_0(x) \\ & \leq \frac{1}{\|h_n^+\|_{L^1(B_{2R}(y_n))}} \int_{B_{2R}(y_n)} \left(\int_{B_R(y_n)} \exp\left(4G(x, y)\|h_n^+\|_{L^1(B_{2R}(y_n))}\right) dV_0(x)\right) h_n^+(y) dV_0(y) \\ & \leq \sup_{y \in B_{2R}(y_n)} \int_{B_R(y_n)} \exp\left(4G(x, y)\|h_n^+\|_{L^1(B_{2R}(y_n))}\right) dV_0(x). \end{aligned} \quad (3.9)$$

It follows from (3.9) and (3.5) that

$$\begin{aligned} & \int_{B_R(y_n)} \exp\left(\int_{B_{2R}(y_n)} 4G(x, y)h_n^+(y)dV_0(y)\right) dV_0(x) \\ & \leq \exp\left(C\|h_n^+\|_{L^1(B_{2R}(y_n))}\right) \sup_{y \in B_{2R}(y_0)} \int_{B_R(y_n)} |x - y|^{-\frac{1}{2\pi^2}\|h_n^+\|_{L^1(B_{2R}(y_n))}} dV_0(x) \\ & \leq e^{\pi^2 C} \sup_{y \in B_{2R}(y_n)} \int_{B_R(y_n)} |x - y|^{-\frac{1}{2\pi^2}\|h_n^+\|_{L^1(B_{2R}(y_n))}} dV_0(x), \end{aligned} \quad (3.10)$$

where we have used (3.3).

It is easy to check that, for any $\alpha \in [0, 4)$, and any $y \in M$, we have

$$\int_{B_R(y_n)} |x - y|^{-\alpha} dV_0(x) \leq \frac{C}{4 - \alpha} R^{4-\alpha}.$$

Since $\|h_n^+\|_{L^1(B_{2R}(y_n))} \leq \pi^2$ by (3.3), it follows by taking $\alpha = \frac{1}{2\pi^2}\|h_n^+\|_{L^1(B_{2R}(y_n))} \leq 1$,

$$\int_{B_R(y_n)} |x - y|^{-\frac{1}{2\pi^2}\|h_n^+\|_{L^1(B_{2R}(y_n))}} dV_0(x) \leq C R^{4-\frac{1}{2\pi^2}\|h_n^+\|_{L^1(B_{2R}(y_n))}}. \quad (3.11)$$

Thus it follows from (3.8), (3.10) and (3.11) that

$$\begin{aligned} & \int_{B_R(y_n)} e^{4(u_n - \bar{u}_n)} dV_0 \leq C R^4 \exp\left(C\|h\|_{L^\infty(M)} - \frac{\log R}{2\pi^2}\|h_n^+\|_{L^1(B_{2R}(y_n))}\right) \\ & \times \exp\left(\sup_{x \in B_R(y_n)} \int_{M \setminus B_{2R}(y_n)} 4G(x, y)h_n^+(y)dV_0(y) - \inf_{z \in B_R(y_n)} \int_M 4G(z, y)h_n^-(y)dV_0(y)\right). \end{aligned} \quad (3.12)$$

On the other hand, using again the representation formula (3.6), we have by using (3.7), for any $x \in B_r(x_n)$,

$$u(x) - \bar{u}_n = \int_M G(x, y)h_n^+(y)dV_0(y) - \int_M G(x, y)h_n^-(y)dV_0(y) - \int_M G(x, y)h(y)dV_0(y)$$

$$\begin{aligned}
&\geq \int_{B_r(x_n)} G(x, y) h_n^+(y) dV_0(y) + \int_{M \setminus B_r(x_n)} G(x, y) h_n^+(y) dV_0(y) - \int_M G(x, y) h_n^-(y) dV_0(y) - C \|h\|_{L^\infty(M)} \\
&\geq \inf_{z \in B_r(x_n)} \int_{B_r(x_n)} G(z, y) h_n^+(y) dV_0(y) + \inf_{z \in B_r(x_n)} \int_{M \setminus B_r(x_n)} G(z, y) h_n^+(y) dV_0(y) \\
&\quad - \int_M G(x, y) h_n^-(y) dV_0(y) - C \|h\|_{L^\infty(M)}. \tag{3.13}
\end{aligned}$$

Since by (3.5) we have $G(z, y) \geq -\frac{1}{8\pi^2} \log r - C$ for any $z, y \in B_r(x_n)$, then it follows from (3.13) that for any $x \in B_r(x_n)$,

$$\begin{aligned}
e^{4(u_n(x) - \bar{u}_n)} &\geq \exp \left(-C \|h\|_{L^\infty(M)} - C \|h_n^+\|_{L^1(M)} - \frac{\log r}{2\pi^2} \|h_n^+\|_{L^1(B_r(x_n))} \right) \\
&\times \exp \left(\inf_{z \in B_r(x_n)} \int_{M \setminus B_r(x_n)} 4G(z, y) h_n^+(y) dV_0(y) \right) \exp \left(- \int_M 4G(x, y) h_n^-(y) dV_0(y) \right). \tag{3.14}
\end{aligned}$$

But from (3.1) and (3.2) we have $\|h_n^+\|_{L^1(M)} \leq \|h\|_{L^1(M)} + o_n(1) \leq C \|h\|_{L^\infty(M)} + o_n(1)$. So it follows from (3.14) on integrating on $B_r(x_n)$ that

$$\begin{aligned}
&\int_{B_r(x_n)} e^{4(u_n - \bar{u}_n)} dV_0 \geq C \exp \left(-C \|h\|_{L^\infty(M)} - \frac{\log r}{2\pi^2} \|h_n^+\|_{L^1(B_r(x_n))} \right) \\
&\times \exp \left(\inf_{z \in B_r(x_n)} \int_{M \setminus B_r(x_n)} 4G(z, y) h_n^+(y) dV_0(y) \right) \int_{B_r(x_n)} \exp \left(- \int_M 4G(x, y) h_n^-(y) dV_0(y) \right) dV_0(x). \tag{3.15}
\end{aligned}$$

But by Jensen inequality we have

$$\begin{aligned}
&\int_{B_r(x_n)} \exp \left(- \int_M 4G(x, y) h_n^-(y) dV_0(y) \right) dV_0(x) \\
&\geq |B_r(x_n)| \exp \left(- \frac{1}{|B_r(x_n)|} \int_{B_r(x_n)} \int_M 4G(x, y) h_n^-(y) dV_0(y) dV_0(x) \right). \tag{3.16}
\end{aligned}$$

Since $|B_r(x_n)| \geq Cr^4$, it follows from (3.15) and (3.16) that

$$\begin{aligned}
&\int_{B_r(x_n)} e^{4(u_n - \bar{u}_n)} dV_0 \geq Cr^4 \exp \left(-C \|h\|_{L^\infty(M)} - \frac{\log r}{2\pi^2} \|h_n^+\|_{L^1(B_r(x_n))} \right) \\
&\times \exp \left(\inf_{z \in B_r(x_n)} \int_{M \setminus B_r(x_n)} 4G(z, y) h_n^+(y) dV_0(y) - \frac{1}{|B_r(x_n)|} \int_{B_r(x_n)} \int_M 4G(x, y) h_n^-(y) dV_0(y) dV_0(x) \right). \tag{3.17}
\end{aligned}$$

Now since

$$\frac{\int_{B_R(y_n)} e^{4u_n} dV_0}{\int_{B_r(x_n)} e^{4u_n} dV_0} = \frac{\int_{B_R(y_n)} e^{4(u_n - \bar{u}_n)} dV_0}{\int_{B_r(x_n)} e^{4(u_n - \bar{u}_n)} dV_0},$$

then it follows from (3.12) and (3.17) that

$$\frac{\int_{B_R(y_n)} e^{4u_n} dV_0}{\int_{B_r(x_n)} e^{4u_n} dV_0} \leq C \left(\frac{r}{R} \right)^{-4} \exp \left(C \|h\|_{L^\infty(M)} + \frac{\log r}{2\pi^2} \|h_n^+\|_{L^1(B_r(x_n))} - \frac{\log R}{2\pi^2} \|h_n^+\|_{L^1(B_{2R}(y_n))} \right)$$

$$\begin{aligned}
& \times \exp \left(\sup_{x \in B_R(y_n)} \int_{M \setminus B_{2R}(y_n)} 4G(x, y) h_n^+(y) dV_0(y) - \inf_{z \in B_r(x_n)} \int_{M \setminus B_r(x_n)} 4G(z, y) h_n^+(y) dV_0(y) \right) \\
& \times \exp \left(\frac{1}{|B_r(x_n)|} \int_{B_r(x_n)} \int_M 4G(x, y) h_n^-(y) dV_0(y) dV_0(x) - \inf_{z \in B_R(y_n)} \int_M 4G(z, y) h_n^-(y) dV_0(y) \right).
\end{aligned} \tag{3.18}$$

Set

$$A = \exp \left(\sup_{x \in B_R(y_n)} \int_{M \setminus B_{2R}(y_n)} 4G(x, y) h_n^+(y) dV_0(y) - \inf_{z \in B_r(x_n)} \int_{M \setminus B_r(x_n)} 4G(z, y) h_n^+(y) dV_0(y) \right)$$

and

$$B = \exp \left(\frac{1}{|B_r(x_n)|} \int_{B_r(x_n)} \int_M 4G(x, y) h_n^-(y) dV_0(y) dV_0(x) - \inf_{z \in B_R(y_n)} \int_M 4G(z, y) h_n^-(y) dV_0(y) \right).$$

We shall prove that

$$A \leq C \exp \left(\frac{\log R}{2\pi^2} \|h_n^+\|_{L^1(B_{2R}(y_n))} - \frac{\log R}{2\pi^2} \|h_n^+\|_{L^1(B_r(x_n))} \right) \tag{3.19}$$

and

$$B \leq C \exp \left(C \|h_n^-\|_{L^1(M)} \log \frac{R}{r} \right). \tag{3.20}$$

It is clear that Proposition 3.1 will follow from (3.18), (3.19) and (3.20) by using (3.2). Let us then prove the estimates (3.19) and (3.20).

First we shall prove (3.19). We have for any $x \in B_R(y_n)$ and $z \in B_r(x_n)$

$$\begin{aligned}
& \int_{M \setminus B_{2R}(y_n)} 4G(x, y) h_n^+(y) dV_0(y) - \int_{M \setminus B_r(x_n)} 4G(z, y) h_n^+(y) dV_0(y) \\
& = \int_{M \setminus B_{4C_0R}(y_n)} 4(G(x, y) - G(z, y)) h_n^+(y) dV_0(y) \\
& + \int_{B_{4C_0R}(y_n) \setminus B_{2R}(y_n)} 4G(x, y) h_n^+(y) dV_0(y) - \int_{B_{4C_0R}(y_n) \setminus B_r(x_n)} 4G(z, y) h_n^+(y) dV_0(y)
\end{aligned} \tag{3.21}$$

since $B_{2R}(y_n) \subset B_{4C_0R}(y_n)$ and $B_r(x_n) \subset B_{4C_0R}(y_n)$, where C_0 is the constant in (3.3) that we assume satisfying $C_0 \geq 1$ without loss of generality (we recall here that $r = r_n \leq R_n = R$).

Let us estimate the first term in the right side of (3.21). We have for any $x \in B_R(y_n)$, $z \in B_r(x_n)$ and $y \in M \setminus B_{4C_0R}(y_n)$ by using the hypothesis $|x_n - y_n| \leq C_0R$ and $r \leq R$, that

$$|x - y| \geq |y - y_n| - |x - y_n| \geq 4C_0R - R \geq 3C_0R, \tag{3.22}$$

and

$$|z - y| \geq |y - y_n| - |y_n - x_n| - |z - x_n| \geq 4C_0R - C_0R - r \geq 2C_0R. \tag{3.23}$$

On the other hand, we have by using (3.22)

$$\begin{aligned}
|z - y| &\leq |z - x_n| + |x_n - y_n| + |y_n - x| + |x - y| \\
&\leq r + C_0 R + R + |x - y| \leq 3C_0 R + |x - y| \\
&\leq 2|x - y|,
\end{aligned}$$

and by using (3.23) we have

$$\begin{aligned}
|x - y| &\leq |x - y_n| + |x_n - y_n| + |x_n - z| + |z - y| \\
&\leq R + C_0 R + r + |z - y| \leq 3C_0 R + |z - y| \\
&\leq \frac{5}{2}|z - y|.
\end{aligned}$$

Thus we have

$$\frac{2}{5} \leq \frac{|z - y|}{|x - y|} \leq 2. \quad (3.24)$$

It follows from (3.5) and (3.24) that for any $x \in B_R(y_n)$, $z \in B_r(x_n)$ and $y \in M \setminus B_{4C_0 R}(y_n)$,

$$|G(x, y) - G(z, y)| \leq C, \quad (3.25)$$

which gives

$$\int_{M \setminus B_{4C_0 R}(y_n)} 4(G(x, y) - G(z, y)) h_n^+(y) dV_0(y) \leq C \|h_n^+\|_{L^1(M)} \quad (3.26)$$

for any $x \in B_R(y_n)$, $z \in B_r(x_n)$.

Now we shall estimate the second and third term in the right side (3.21). By using formula (3.5) we have for any $x \in B_R(y_n)$ and $y \in B_{4C_0 R}(y_n) \setminus B_{2R}(y_n)$,

$$G(x, y) \leq -\frac{1}{8\pi^2} \log R + C. \quad (3.27)$$

We have for any $z \in B_r(x_n)$ and $y \in B_{4C_0 R}(y_n) \setminus B_r(x_n)$, since $|x_n - y_n| \leq C_0 R$ (by hypothesis) and $r \leq R$,

$$|y - z| \leq |y - y_n| + |y_n - x_n| + |x_n - z| \leq 4C_0 R + C_0 R + r \leq 6C_0 R$$

which implies by (3.5) that

$$G(z, y) \geq -\frac{1}{8\pi^2} \log R + C. \quad (3.28)$$

It follows from (3.27) and (3.28), for any $x \in B_R(y_n)$ and $z \in B_r(x_n)$, that

$$\begin{aligned}
&\int_{B_{4C_0 R}(y_n) \setminus B_{2R}(y_n)} 4G(x, y) h_n^+(y) dV_0(y) - \int_{B_{4C_0 R}(y_n) \setminus B_r(x_n)} 4G(z, y) h_n^+(y) dV_0(y) \\
&\leq \frac{\log R}{2\pi^2} \int_{B_{2R}(y_n)} h_n^+ dV_0 - \frac{\log R}{2\pi^2} \int_{B_r(x_n)} h_n^+ dV_0 + C \|h_n^+\|_{L^1(M)}.
\end{aligned} \quad (3.29)$$

Combining (3.21), (3.26) and (3.29) we obtain the desired estimate (3.19) since $\|h_n^+\|_{L^1(M)} \leq \|h\|_{L^1(M)} + \|h_n^-\|_{L^1(M)} \leq C$ by integrating (3.1) and using (3.2).

Now it remains to prove (3.20). We have

$$B = \exp \left\{ \frac{1}{|B_r(x_n)|} \int_{B_r(x_n)} \int_M 4G(x, y) h_n^-(y) dV_0(y) dV_0(x) - \inf_{z \in B_R(y_n)} \int_M 4G(z, y) h_n^-(y) dV_0(y) \right\}$$

$$= \sup_{z \in B_R(y_n)} \exp \left\{ \frac{1}{|B_r(x_n)|} \int_{B_r(x_n)} \left(\int_M 4(G(x, y) - G(z, y)) h_n^-(y) dV_0(y) \right) dV_0(x) \right\}. \quad (3.30)$$

But we have for any $z \in B_R(y_n)$,

$$\begin{aligned} & \frac{1}{|B_r(x_n)|} \int_{B_r(x_n)} \left(\int_M 4(G(x, y) - G(z, y)) h_n^-(y) dV_0(y) \right) dV_0(x) \\ &= \frac{1}{|B_r(x_n)|} \int_{B_r(x_n)} \left(\int_{B_{4C_0R}(y_n)} 4(G(x, y) - G(z, y)) h_n^-(y) dV_0(y) \right) dV_0(x) \\ &+ \frac{1}{|B_r(x_n)|} \int_{B_r(x_n)} \left(\int_{M \setminus B_{4C_0R}(y_n)} 4(G(x, y) - G(z, y)) h_n^-(y) dV_0(y) \right) dV_0(x). \end{aligned} \quad (3.31)$$

Since by (3.5) we have, for any $z \in B_R(y_n)$ and $y \in B_{4C_0R}(y_n)$,

$$G(z, y) \geq -\frac{1}{8\pi^2} \log R + C,$$

then the first term in the right side of (3.31) can be estimated as follows

$$\begin{aligned} & \frac{1}{|B_r(x_n)|} \int_{B_r(x_n)} \int_{B_{4C_0R}(y_n)} 4(G(x, y) - G(z, y)) h_n^-(y) dV_0(y) dV_0(x) \\ &\leq \frac{1}{|B_r(x_n)|} \int_{B_r(x_n)} \int_{B_{4C_0R}(y_n)} \frac{1}{8\pi^2} (-\log |x - y| + \log R) h_n^-(y) dV_0(y) dV_0(x) + C \|h_n^-\|_{L^1(M)} \\ &\leq \frac{1}{8\pi^2} \|h_n^-\|_{L^1(M)} \frac{1}{|B_r(x_n)|} \sup_{y \in B_{4C_0R}(x_n)} \int_{B_r(x_n)} \left| \log \left(\frac{1}{R} |x - y| \right) \right| dV_0(x) + C \|h_n^-\|_{L^1(M)}. \end{aligned} \quad (3.32)$$

A direct computation shows that

$$\frac{1}{|B_r(x_n)|} \sup_{y \in B_{4C_0R}(y_n)} \int_{B_r(x_n)} \left| \log \left(\frac{1}{R} |x - y| \right) \right| dV_0(x) \leq C \log \left(\frac{R}{r} \right) + C.$$

(we recall here that $r \leq R$). Combining the last inequality with (3.32) gives for any $z \in B_R(y_n)$,

$$\begin{aligned} & \frac{1}{|B_r(x_n)|} \int_{B_r(x_n)} \int_{B_{4C_0R}(y_n)} 4(G(x, y) - G(z, y)) h_n^-(y) dV_0(y) dV_0(x) \\ &\leq C \|h_n^-\|_{L^1(M)} \log \left(\frac{R}{r} \right) + C \|h_n^-\|_{L^1(M)}. \end{aligned} \quad (3.33)$$

Now let us estimate the second term in the right side of (3.31). We recall that from (3.25) we have

$$|G(x, y) - G(z, y)| \leq C,$$

for any $x \in B_r(x_n)$, $z \in B_R(y_n)$ and $y \in M \setminus B_{4C_0R}(y_n)$. Thus we obtain

$$\frac{1}{|B_r(x_n)|} \int_{B_r(x_n)} \int_{M \setminus B_{4C_0R}(y_n)} 4(G(x, y) - G(z, y)) h_n^-(y) dV_0(y) dV_0(x) \leq C \|h_n^-\|_{L^1(M)}. \quad (3.34)$$

It follows from (3.30), (3.31), (3.33) and (3.34) that

$$B \leq \exp \left(C \|h_n^-\|_{L^1(M)} \log \left(\frac{R}{r} \right) + C \|h_n^-\|_{L^1(M)} \right),$$

which proves (3.20) by using (3.2). The proof of Proposition 3.1 is then complete. \square

4. VOLUME QUANTIZATION AND PROOF OF THEOREM 1.1

In this section we apply the result of section 3 (Harnack type inequality) to prove some fundamental properties on solutions of equation (1.8) in Theorem 1.1. They state that the conformal volume concentrates with quantization at points corresponding to blow-up sequences. Through the rest of the paper we shall assume that $k_0 = \int_M f dV_0 > 0$ where f is as in Theorem 1.1. Indeed, if $k_0 \leq 0$, then Theorem 1.1 is a direct consequence of Proposition 2.3 in section 2.

Proposition 4.1. *Let $(u_n)_n$ as in Theorem 1.1 and let $(x_n, r_n)_n$ a blow-up for the sequence $(u_n)_n$. Let $(y_n)_n$ a sequence of points in M , and $0 < \rho_n \leq \text{diam}(M)$ such that $\lim_{n \rightarrow +\infty} \frac{r_n}{\rho_n} = 0$. Suppose that, for some positive constant C_0 independent of n , we have*

$$|x_n - y_n| \leq C_0 \rho_n \quad \text{and} \quad \int_{B_{2\rho_n}(y_n)} e^{4u_n} dV_0 \leq \frac{\pi^2}{k_0},$$

where $k_0 = \int_M f dV_0$. Then

$$\int_{B_{\rho_n}(y_n)} e^{4u_n} dV_0 \leq C \left(\frac{r_n}{\rho_n} \right)^{2+o_n(1)}, \quad (4.1)$$

where C is a positive constant independent of n . In particular, we have

$$\lim_{n \rightarrow +\infty} \int_{B_{\rho_n}(y_n)} e^{4u_n} dV_0 = 0.$$

Proof. First let us apply Proposition 2.4 by choosing $\beta_n = \sqrt{\frac{\rho_n}{r_n}}$. Then there exists $b_n \leq \sqrt{\frac{\rho_n}{r_n}}$ such that $b_n \rightarrow +\infty$ and

$$\lim_{n \rightarrow +\infty} \int_{B_{b_n r_n}(x_n)} e^{4u_n} dV_0 = \frac{16\pi^2}{k_0}. \quad (4.2)$$

We can apply now Proposition 3.1 to $(u_n)_n$ by choosing $h_n = k_n e^{4u_n} - f_n + f$, $h = f$, $R_n = \rho_n$, and $b_n r_n$ instead of r_n . Indeed, since $f_n \xrightarrow[n \rightarrow +\infty]{} f$ in $L^1(M)$ and $k_0 = \int_M f dV_0 > 0$, then $k_n = \int_M f_n dV_0 > 0$ for n large enough. Then one can easily check that hypothesis (3.2)-(3.3) in Proposition 3.1 are satisfied. Thus we obtain

$$\begin{aligned} \int_{B_{\rho_n}(y_n)} e^{4u_n} dV_0 &\leq C \left(\frac{b_n r_n}{\rho_n} \right)^{-4 + \frac{1}{2\pi^2} \|h_n^+\|_{L^1(B_{b_n r_n}(x_n))} + o_n(1)} \int_{B_{b_n r_n}(x_n)} e^{4u_n} dV_0 \\ &\leq C \left(\frac{b_n r_n}{\rho_n} \right)^{-4 + \frac{1}{2\pi^2} \|h_n^+\|_{L^1(B_{b_n r_n}(x_n))} + o_n(1)}, \end{aligned} \quad (4.3)$$

where we have used the fact that $\int_{B_{b_n r_n}(x_n)} e^{4u_n} dV_0 \leq \int_M e^{4u_n} dV_0 = 1$.

Since $h_n = k_n e^{4u_n} - f_n + f$, then we have by using (4.2) and the fact that $f_n \rightarrow f$ in $L^1(M)$, that

$$\frac{1}{2\pi^2} \|h_n^+\|_{L^1(B_{b_n r_n}(x_n))} = 8 + o_n(1)$$

and by replacing in (4.3) we get

$$\int_{B_{\rho_n}(y_n)} e^{4u_n} dV_0 \leq C \left(\frac{b_n r_n}{\rho_n} \right)^{4+o_n(1)}.$$

This proves estimate (4.1) since $b_n \leq \sqrt{\frac{\rho_n}{r_n}}$.

□

Proposition 4.2. *Let $(u_n)_n$ as in Theorem 1.1 and let $(x_n, r_n)_n$ a blow-up for the sequence $(u_n)_n$. Let $0 < R_n \leq S_n$ such that $\frac{r_n}{R_n} \xrightarrow{n \rightarrow +\infty} 0$, and suppose that there exists a positive constant $\alpha \leq 1$ independent of n such that*

$$\forall B_r(y) \subset B_{2S_n}(x_n) \setminus B_{\frac{1}{2}R_n}(x_n), \int_{B_r(y)} e^{4u_n} dV_0 \geq \frac{\pi^2}{k_0} \implies r \geq \alpha |y - x_n|. \quad (4.4)$$

Then

$$\lim_{n \rightarrow +\infty} \int_{B_{S_n}(x_n) \setminus B_{R_n}(x_n)} e^{4u_n} dV_0 = 0. \quad (4.5)$$

Proof. Before giving the proof we note here that we may assume without loss of generality that $S_n \leq \text{diam}(M)$. First we shall prove that for any $\rho_n \in [R_n, S_n]$ we have the following estimate

$$\int_{B_{\frac{3}{2}\rho_n}(x_n) \setminus B_{\rho_n}(x_n)} e^{4u_n} dV_0 \leq C \left(\frac{r_n}{\rho_n} \right)^{2+o_n(1)}, \quad (4.6)$$

where C is constant independent of n , and $o_n(1) \rightarrow 0$ as $n \rightarrow +\infty$. Then (4.5) will follow from (4.6) by using an appropriate decomposition of the annulus $B_{S_n}(x_n) \setminus B_{R_n}(x_n)$. Indeed, suppose that (4.6) is proved, then by choosing $N \in \mathbb{N}$ such that $(3/2)^N \leq S_n/R_n \leq (3/2)^{N+1}$, and applying (4.6) with $\rho_n = (3/2)^j R_n$ for $j = 0, \dots, N$, we obtain for n large enough :

$$\int_{B_{(3/2)^{j+1}R_n}(x_n) \setminus B_{(3/2)^j R_n}(x_n)} e^{4u_n} dV_0 \leq C(2/3)^{(2+o_n(1))j} \left(\frac{r_n}{R_n} \right)^{2+o_n(1)} \leq C(2/3)^j \left(\frac{r_n}{R_n} \right)^{2+o_n(1)},$$

and by summing up over $j = 0, \dots, N$, one gets

$$\begin{aligned} \int_{B_{S_n}(x_n) \setminus B_{R_n}(x_n)} e^{4u_n} dV_0 &\leq \sum_{j=0}^N \int_{B_{(3/2)^{j+1}R_n}(x_n) \setminus B_{(3/2)^j R_n}(x_n)} e^{4u_n} dV_0 \leq \\ &C \left(\frac{r_n}{R_n} \right)^{2+o_n(1)} \sum_{j=0}^N (2/3)^j \leq 3C \left(\frac{r_n}{R_n} \right)^{2+o_n(1)} \rightarrow 0 \text{ as } n \rightarrow +\infty, \end{aligned}$$

which proves the desired result (4.5).

Now let us prove the estimate (4.6). We can cover the set $B_{\frac{3}{2}\rho_n}(x_n) \setminus B_{\rho_n}(x_n)$ by a finite number of balls $B_{\frac{1}{4}\alpha\rho_n}(z_1), \dots, B_{\frac{1}{4}\alpha\rho_n}(z_L)$, where $L \in \mathbb{N}$ is independent of n , and where α is the constant appearing in (4.4), such that

$$B_{\frac{1}{2}\alpha\rho_n}(z_j) \subset B_{2\rho_n}(x_n) \setminus B_{\frac{1}{2}\rho_n}(x_n) \subset B_{2S_n}(x_n) \setminus B_{\frac{1}{2}R_n}(x_n), \quad j = 1, \dots, L.$$

But since $|z_j - x_n| \geq \rho_n > \frac{1}{2}\rho_n$, we have from (4.4) that

$$\int_{B_{\frac{1}{2}\alpha\rho_n}(z_j)} e^{4u_n} dV_0 < \frac{\pi^2}{k_0} \quad \forall j = 1, \dots, L. \quad (4.7)$$

(we recall here that $0 < \alpha \leq 1$.)

We can now apply Proposition 4.1 by taking $y_n = z_j$ to get

$$\int_{B_{\frac{1}{4}\alpha\rho_n}(z_j)} e^{4u_n} dV_0 \leq C \left(\frac{r_n}{\rho_n} \right)^{2+o_n(1)} \quad \forall j = 1, \dots, L,$$

and the estimate (4.6) follows. This achieves the proof of Proposition 4.2. \square

Proposition 4.3. *Let (u_n, f_n) as in Theorem 1.1. Let $(x_n^1, r_n^1)_n, \dots, (x_n^m, r_n^m)_n$ be m blow-ups for $(u_n)_n$, and $R_n^1, \dots, R_n^m > 0$ such that*

$$\lim_{n \rightarrow +\infty} R_n^i = 0, \quad \lim_{n \rightarrow +\infty} \frac{r_n^i}{R_n^i} = 0 \quad \forall i = 1, \dots, m, \quad (4.8)$$

and

$$\lim_{n \rightarrow +\infty} \frac{R_n^i}{|x_n^j - x_n^i|} = 0 \quad \forall i \neq j \text{ in } \{1, \dots, m\} \text{ if } m \geq 2. \quad (4.9)$$

Let $S_n \geq 4 \max_{i \neq j} |x_n^i - x_n^j|$ and suppose that there exists a positive constant $\alpha \leq 1$ independent of n such that

$$\forall B_r(y) \subset \bigcup_{j=1}^m B_{2S_n}(x_n^j) \setminus \bigcup_{j=1}^m B_{\frac{1}{2}R_n^j}(x_n^j), \quad \int_{B_r(y)} e^{4u_n} dV_0 \geq \frac{\pi^2}{k_0} \implies r \geq \alpha d_n(y), \quad (4.10)$$

where $d_n(y) = \inf_{1 \leq j \leq m} |y - x_n^j|$. Then

$$\lim_{n \rightarrow +\infty} \int_{\bigcup_{j=1}^m B_{S_n}(x_n^j) \setminus \bigcup_{j=1}^m B_{R_n^j}(x_n^j)} e^{4u_n} dV_0 = 0. \quad (4.11)$$

Proof. It is clear that it suffices to prove (4.11) for a subsequence of $(u_n)_n$. We proceed by induction on m . Suppose $m = 1$, then it follows from Proposition 4.2, by taking $x_n = x_n^1$ and $R_n = R_n^1$, that

$$\lim_{n \rightarrow +\infty} \int_{B_{S_n}(x_n^1) \setminus B_{R_n^1}(x_n^1)} e^{4u_n} dV_0 = 0.$$

Now let $m \geq 2$ be an integer and suppose that (4.11) is true for any l blow-ups with $l \leq m$. We shall prove that this is also the case for any $(m+1)$ blow-ups. Let then $(x_n^1, r_n^1)_n, \dots, (x_n^{m+1}, r_n^{m+1})_n$

be $(m+1)$ blow-ups for $(u_n)_n$ satisfying (4.9)-(4.10) for some $R_n^i > 0$, $i \in \llbracket 1, m+1 \rrbracket$ and $S_n > 0$, that is

$$\lim_{n \rightarrow +\infty} \frac{r_n^i}{R_n^i} = 0 \quad \forall i = 1, \dots, m+1, \quad \lim_{n \rightarrow +\infty} \frac{R_n^i}{|x_n^i - x_n^j|} = 0 \quad \forall i \neq j \text{ in } \{1, \dots, m+1\}, \quad (4.12)$$

and

$$\forall B_r(y) \subset \bigcup_{j=1}^{m+1} B_{2S_n}(x_n^j) \setminus \bigcup_{j=1}^{m+1} B_{\frac{1}{2}R_n^j}(x_n^j), \quad \int_{B_r(y)} e^{4u_n} dV_0 \geq \frac{\pi^2}{k_0} \implies r \geq \alpha d_n(y), \quad (4.13)$$

where $d_n(y) = \inf_{1 \leq j \leq m+1} |y - x_n^j|$.

Let

$$d_n = \inf \{ |x_n^i - x_n^j| : i, j \in \llbracket 1, m+1 \rrbracket, i \neq j \}$$

and

$$D_n = \sup \{ |x_n^i - x_n^j| : i, j \in \llbracket 1, m+1 \rrbracket, i \neq j \}.$$

By passing to a subsequence if necessary, we distinguish two cases depending on d_n and D_n . That is, we have either $D_n \leq Cd_n$, where C is a positive constant independent of n , or $\lim_{n \rightarrow +\infty} \frac{d_n}{D_n} = 0$.

First case : $D_n \leq Cd_n$, where C is a positive constant independent of n .

If we apply Proposition 4.2 by taking $x_n = x_n^i$ and $R_n = 4D_n$ (by using (4.13)), we have for any $i = 1, \dots, m+1$

$$\lim_{n \rightarrow +\infty} \int_{B_{S_n}(x_n^i) \setminus B_{4D_n}(x_n^i)} e^{4u_n} dV_0 = 0.$$

Thus it remains to prove that

$$\lim_{n \rightarrow +\infty} \int_{\bigcup_{j=1}^{m+1} B_{4D_n}(x_n^j) \setminus \bigcup_{j=1}^{m+1} B_{R_n^j}(x_n^j)} e^{4u_n} dV_0 = 0. \quad (4.14)$$

We have by (4.12) since $D_n \leq Cd_n$ that $\lim_{n \rightarrow \infty} \frac{R_n^j}{d_n} = 0$, $j = 1, \dots, m+1$. Thus if we apply Proposition 4.2 by taking $x_n = x_n^j$, $R_n = R_n^j$ and $S_n = \frac{1}{4}d_n$ (by using (4.13)), we obtain

$$\lim_{n \rightarrow +\infty} \int_{B_{\frac{1}{4}d_n}(x_n^j) \setminus B_{R_n^j}(x_n^j)} e^{4u_n} dV_0 = 0 \quad \forall j = 1, \dots, m+1. \quad (4.15)$$

On the other hand, since $d_n \leq D_n \leq Cd_n$, we can cover the set $\bigcup_{j=1}^{m+1} B_{4D_n}(x_n^j) \setminus \bigcup_{j=1}^{m+1} B_{\frac{1}{4}d_n}(x_n^j)$ by a finite number N (independent of n) of balls $B_{\frac{1}{16}\alpha d_n}(z_n^l)$, $l = 1, \dots, N$, where $0 < \alpha \leq 1$ is the constant appearing in (4.13), such that $B_{\frac{1}{8}\alpha d_n}(z_n^l) \subset \bigcup_{j=1}^{m+1} B_{2D_n}(x_n^j) \setminus \bigcup_{j=1}^{m+1} B_{\frac{1}{8}d_n}(x_n^j)$. Then we can apply Proposition 4.1 by taking $y_n = z_n^l$, $x_n = x_n^1$, $r_n = r_n^1$, $\rho_n = \frac{1}{16}\alpha d_n$, and by using (4.13), we obtain

$$\lim_{n \rightarrow +\infty} \int_{B_{\frac{1}{16}\alpha d_n}(z_n^l)} e^{4u_n} dV_0 = 0 \quad \forall l = 1, \dots, N. \quad (4.16)$$

It is clear that (4.14) follows from (4.15) and (4.16).

Second case : $\lim_{n \rightarrow +\infty} \frac{d_n}{D_n} = 0.$

By relabelling the blow-ups and passing to a subsequence if necessary, we may suppose that $d_n = |x_n^1 - x_n^2|$. Define the set J by :

$$J = \{ j \in \llbracket 1, m+1 \rrbracket : |x_n^j - x_n^1| \leq C_j d_n \ \forall n \} ,$$

where C_j is a positive constant independent of n . By Taking $C_0 = \max_{j \in J} C_j$ we have (by passing to a subsequence if necessary)

$$\forall j \in J, |x_n^j - x_n^1| \leq C_0 d_n \ \forall n, \quad (4.17)$$

and

$$\forall j \in \llbracket 1, m+1 \rrbracket \setminus J, \lim_{n \rightarrow +\infty} \frac{d_n}{|x_n^j - x_n^1|} = 0. \quad (4.18)$$

By relabeling the blow-ups (except for $j = 1$ and $j = 2$) and observing that $1, 2 \in J$, we may suppose that $J = \llbracket 1, k \rrbracket$, where k satisfies $2 \leq k \leq m$ since $\frac{d_n}{D_n} \xrightarrow{n \rightarrow +\infty} 0$. Now by using (4.12)-(4.13) and (4.17)-(4.18), we can apply the induction hypothesis above to the k blow-ups: $(x_n^1, r_n^1), \dots, (x_n^k, r_n^k)$, where S_n is replaced by $\tilde{S}_n = 8C_0 d_n$, and where C_0 is the constant in (4.17). We obtain

$$\lim_{n \rightarrow +\infty} \int_{\bigcup_{j=1}^k B_{8C_0 d_n}(x_n^j) \setminus \bigcup_{j=1}^k B_{R_n^j}(x_n^j)} e^{4u_n} dV_0 = 0. \quad (4.19)$$

On the other hand, for each fixed $i \in \llbracket 1, k \rrbracket$, if we apply again the induction hypothesis to the $(m+2-k)$ blow-ups : $x_n^i, x_n^{k+1}, x_n^{k+2}, \dots, x_n^{m+1}$ (we recall here that $2 \leq k \leq m$) where R_n^i is replaced by $\tilde{R}_n^i = 8C_0 d_n$, then we have for any $i \in \llbracket 1, k \rrbracket$,

$$\lim_{n \rightarrow +\infty} \int_{\left(\bigcup_{j=k+1}^{m+1} B_{S_n}(x_n^j) \cup B_{S_n}(x_n^i) \right) \setminus \left(\bigcup_{j=k+1}^{m+1} B_{R_n^j}(x_n^j) \cup B_{8C_0 d_n}(x_n^i) \right)} e^{4u_n} dV_0 = 0$$

which gives

$$\lim_{n \rightarrow +\infty} \int_{\bigcup_{j=1}^{m+1} B_{S_n}(x_n^j) \setminus \left(\bigcup_{j=k+1}^{m+1} B_{R_n^j}(x_n^j) \cup \bigcup_{i=1}^k B_{8C_0 d_n}(x_n^i) \right)} e^{4u_n} dV_0 = 0. \quad (4.20)$$

Now it is clear that (4.19) and (4.20) imply

$$\lim_{n \rightarrow +\infty} \int_{\bigcup_{j=1}^{m+1} B_{S_n}(x_n^j) \setminus \bigcup_{j=1}^{m+1} B_{R_n^j}(x_n^j)} e^{4u_n} dV_0 = 0.$$

This achieves the proof of Proposition 4.3. □

The following proposition is the principal tool in the proof of Theorem 1.1

Proposition 4.4. *Let (u_n, f_n) as in Theorem 1.1. If the first alternative in Theorem 1.1 does not hold, then there exist a finite number of blow-ups $(x_n^1, r_n^1)_n, \dots, (x_n^k, r_n^k)_n$ with $1 \leq k \leq \frac{k_0}{16\pi^2}$, and k sequences $(R_n^1)_n, \dots, (R_n^k)_n$ of positive numbers such that*

$$\lim_{n \rightarrow +\infty} R_n^i = 0 \quad , \quad \lim_{n \rightarrow +\infty} \frac{r_n^i}{R_n^i} = 0 \quad \forall i \in \llbracket 1, k \rrbracket, \quad (4.21)$$

$$\lim_{n \rightarrow +\infty} \frac{R_n^i}{\inf_{\substack{1 \leq j \leq k \\ j \neq i}} |x_n^i - x_n^j|} = 0, \quad \forall i \in \llbracket 1, k \rrbracket \quad \text{if } k \geq 2, \quad (4.22)$$

and

$$\forall B_r(y) \subset M \setminus \bigcup_{j=1}^k B_{\frac{1}{2}R_n^j}(x_n^j), \quad \int_{B_r(y)} e^{4u_n} dV_0 \geq \frac{\pi^2}{k_0} \implies r \geq \alpha d_n(y), \quad (4.23)$$

where α is a positive constant independent of n , and where $d_n(y) = \inf_{1 \leq j \leq k} |y - x_n^j|$. Moreover we have, for all $i \in \llbracket 1, k \rrbracket$,

$$\lim_{n \rightarrow +\infty} \int_{B_{R_n^i}(x_n^i)} e^{4u_n} dV_0 = \frac{16\pi^2}{k_0}. \quad (4.24)$$

Proof. Before proving Proposition 4.4 let us introduce some notations. If $(x_n^1, r_n^1)_n, \dots, (x_n^l, r_n^l)_n$ are l blow-ups for $(u_n)_n$, we say that they satisfy the property (\mathcal{P}) if

$$l = 1 \quad \text{or} \quad \lim_{n \rightarrow +\infty} \frac{r_n^i}{\inf_{\substack{1 \leq j \leq l \\ j \neq i}} |x_n^i - x_n^j|} = 0, \quad \forall i \in \llbracket 1, l \rrbracket \quad \text{if } l \geq 2.$$

Now let us prove Proposition 4.4. As noted in the begining of this section, we may suppose that $k_0 > 0$. If the first alternative in Theorem 1.1 does not hold, then by using Proposition 2.3, there exists a point $x \in M$ such that for any $r > 0$ we have

$$\liminf_{n \rightarrow +\infty} \int_{B_r(x)} e^{4u_n} dV_0 \geq \frac{8\pi^2}{k_0} + o_r(1).$$

where $o_r(1) \rightarrow 0$ as $r \rightarrow 0$. It follows that there exist $x_n \in M$ and $r_n > 0$ such that

$$\frac{\pi^2}{k_0} = \int_{B_{r_n}(x_n)} e^{4u_n} dV_0 = \sup_{x \in M} \int_{B_{r_n}(x)} e^{4u_n} dV_0 \quad \text{and} \quad \lim_{n \rightarrow +\infty} r_n = 0.$$

Then setting $\hat{r}_n = \sqrt{r_n}$ we have that for any $y \in B_{\hat{r}_n}(x_n)$, $\int_{B_{\hat{r}_n}(y)} e^{4u_n} dV_0 < \frac{\pi^2}{k_0}$, and applying Proposition 2.5, we see that (x_n, r_n) is a blow-up for $(u_n)_n$. It follows that the set A defined by

$$A := \{l \in \mathbb{N} : \text{there exist } l \text{ blow-ups } (x_n^1, r_n^1)_n, \dots, (x_n^l, r_n^l)_n \text{ satisfying the property } (\mathcal{P})\}$$

is not empty. First we shall prove that if $l \in A$, then $l \leq \frac{16\pi^2}{k_0}$. Indeed, let $l \in A$. Then there exist l blow-ups $(x_n^1, r_n^1)_n, \dots, (x_n^l, r_n^l)_n$ satisfying the property (\mathcal{P}) above. More precisely, we have

$$\lim_{n \rightarrow +\infty} \frac{r_n^i}{d_n^i} = 0 \quad \forall i \in \llbracket 1, l \rrbracket, \quad (4.25)$$

where $d_n^i = \inf_{\substack{1 \leq j \leq l \\ j \neq i}} |x_n^i - x_n^j|$ if $l \geq 2$, and $d_n^1 = 1$ if $l = 1$.

Now we apply Proposition 2.3 by setting $\beta_n = \frac{d_n^i}{4r_n^i}$. Then there exists $(b_n^i)_n$ satisfying $b_n^i \leq \beta_n$ and $b_n^i \xrightarrow{n \rightarrow +\infty} +\infty$, such that

$$\lim_{n \rightarrow +\infty} \int_{B_{b_n^i r_n^i}(x_n^i)} e^{4u_n} dV_0 = \frac{16\pi^2}{k_0}. \quad (4.26)$$

Since $b_n^i r_n^i \leq \frac{1}{4}d_n^i$, then the balls $B_{b_n^i r_n^i}(x_n^i)$, $i = 1, \dots, l$, are pairwise disjoint. This implies by using (4.26) that

$$\frac{16\pi^2 l}{k_0} = \lim_{n \rightarrow +\infty} \int_{\bigcup_{i=1}^l B_{b_n^i r_n^i}(x_n^i)} e^{4u_n} dV_0 \leq \int_M e^{4u_n} dV_0 = 1,$$

which implies $l \leq \frac{k_0}{16\pi^2}$. Hence the set A defined above is bounded, so let $k := \max A$. Thus, there exist k blow-ups $(x_n^1, r_n^1)_n, \dots, (x_n^k, r_n^k)_n$ satisfying the property (\mathcal{P}) defined above. That is,

$$\lim_{n \rightarrow +\infty} \frac{r_n^i}{d_n^i} = 0 \quad \forall i \in \llbracket 1, k \rrbracket, \quad (4.27)$$

where $d_n^i = \inf_{\substack{1 \leq j \leq l \\ j \neq i}} |x_n^i - x_n^j|$ if $k \geq 2$, and $d_n^1 = 1$ if $k = 1$. Now, by setting $\beta_n = \frac{1}{2} \sqrt{\frac{d_n^i}{r_n^i}}$ and applying

Proposition 2.3, then there exists $(b_n^i)_n$ satisfying $b_n^i \leq \frac{1}{2} \sqrt{\frac{d_n^i}{r_n^i}}$ and $b_n^i \xrightarrow{n \rightarrow +\infty} +\infty$, such that

$$\lim_{n \rightarrow +\infty} \int_{B_{b_n^i r_n^i}(x_n^i)} e^{4u_n} dV_0 = \frac{16\pi^2}{k_0}, \quad \forall i = 1, \dots, k. \quad (4.28)$$

If we set $R_n^i = b_n^i r_n^i$, then it is clear that (4.21) is satisfied, and (4.24) follows from (4.28). If we apply again Proposition 2.3 by choosing $\beta_n = \frac{1}{4} \frac{R_n^i}{r_n^i}$, and using (4.28) we arrive at

$$\lim_{n \rightarrow +\infty} \int_{B_{R_n^i}(x_n^i) \setminus B_{\frac{1}{4}R_n^i}(x_n^i)} e^{4u_n} dV_0 = 0, \quad \forall i = 1, \dots, k. \quad (4.29)$$

Hence

$$\lim_{n \rightarrow +\infty} \int_{B_{\frac{1}{4}R_n^i}(x_n^i)} e^{4u_n} dV_0 = \lim_{n \rightarrow +\infty} \int_{B_{R_n^i}(x_n^i)} e^{4u_n} dV_0 = \frac{16\pi^2}{k_0}, \quad \forall i = 1, \dots, k. \quad (4.30)$$

Now, Since $R_n^i = b_n^i r_n^i \leq \sqrt{r_n^i d_n^i}$, then we have

$$\lim_{n \rightarrow +\infty} \frac{R_n^i}{d_n^i} = 0 \quad \forall i \in \llbracket 1, k \rrbracket, \quad (4.31)$$

which proves (4.22).

It remains then to prove (4.23). Suppose by contradiction that (4.23) is false, then there are balls $B_{\rho_n}(z_n) \subset M \setminus \bigcup_{j=1}^k B_{\frac{1}{2}R_n^j}(x_n^j)$ such that

$$\int_{B_{\rho_n}(z_n)} e^{4u_n} dV_0 \geq \frac{\pi^2}{k_0} \quad \text{and} \quad \lim_{n \rightarrow +\infty} \frac{\rho_n}{d_n(z_n)} = 0,$$

where we recall that $d_n(z) = \inf_{1 \leq j \leq k} |x_n^j - z|$. Then there exist $r_n \leq \rho_n$ and a ball $B_{r_n}(a_n) \subset M \setminus \bigcup_{j=1}^k B_{\frac{1}{2}R_n^j}(x_n^j)$ such that

$$\frac{\pi^2}{k_0} = \int_{B_{r_n}(a_n)} e^{4u_n} dV_0 = \sup_{B_{r_n}(y) \subset M \setminus \bigcup_{j=1}^k B_{\frac{1}{2}R_n^j}(x_n^j)} \int_{B_{r_n}(y)} e^{4u_n} dV_0. \quad (4.32)$$

Let us show that

$$\lim_{n \rightarrow +\infty} \frac{r_n}{d_n(a_n)} = 0. \quad (4.33)$$

If (4.33) was false, then by passing to a subsequence if necessary, we would have for some constant C independent of n ,

$$r_n \geq C d_n(a_n) \quad (4.34)$$

and without loss of generality we may suppose that $d_n(a_n) = |x_n^1 - a_n|$. Set $d_n := d_n(a_n)$ and define :

$$J := \{ j \in \llbracket 1, k \rrbracket : |x_n^j - x_n^1| \leq C d_n \ \forall n \},$$

where C is a positive constant independent of n . Observing that $1 \in J$, so by relabeling the blow-ups, we may suppose that $J = \llbracket 1, m \rrbracket$, with $1 \leq m \leq k$, and by passing to a subsequence if necessary, we have

$$\forall j \in \llbracket 1, m \rrbracket, |x_n^j - x_n^1| \leq C_0 d_n \ \forall n, \quad (4.35)$$

and

$$\forall j \in \llbracket m+1, k \rrbracket, \lim_{n \rightarrow +\infty} \frac{d_n}{|x_n^j - x_n^1|} = 0, \quad (4.36)$$

where C_0 is a positive constant independent of n that we assume, without loss of generality, satisfying $C_0 \geq 1$.

Now by using (4.32), (4.34) and (4.35) one can easily check that

$$\forall B_r(y) \subset \bigcup_{j=1}^m B_{8C_0 d_n}(x_n^j) \setminus \bigcup_{j=1}^m B_{\frac{1}{2}R_n^j}(x_n^j), \quad \int_{B_r(y)} e^{4u_n} dV_0 \geq \frac{\pi^2}{k_0} \implies r \geq \alpha \inf_{1 \leq j \leq m} |y - x_n^j|, \quad (4.37)$$

where α is a positive constant independent of n . Thus by applying Proposition 4.3, where we take $S_n = 4C_0 d_n$ and using (4.37), we get

$$\lim_{n \rightarrow +\infty} \int_{B_{4C_0 d_n}(x_n^1) \setminus \bigcup_{j=1}^m B_{R_n^j}(x_n^j)} e^{4u_n} dV_0 = 0,$$

which contradicts (4.32) since $B_{r_n}(a_n) \subset B_{4C_0 d_n}(x_n^1) \setminus \bigcup_{j=1}^m B_{R_n^j}(x_n^j)$. So this proves (4.33).

Since $\frac{1}{2}R_n^i \leq |x_n^i - a_n|$, then it follows from (4.29), (4.32) and (4.33) that for n large enough we have

$$R_n^i \leq \frac{4}{3}|x_n^i - a_n| \quad \forall i = 1, \dots, k. \quad (4.38)$$

Indeed, if (4.38) were not satisfied, then by passing to a subsequence one could check by using (4.33) that $B_{r_n}(a_n) \subset B_{R_n^i}(x_n^i) \setminus B_{\frac{1}{4}R_n^i}(x_n^i)$, so by (4.29) we would have that $\lim_{n \rightarrow +\infty} \int_{B_{r_n}(a_n)} e^{4u_n} dV_0 = 0$ contradicting thus (4.32).

Now, by using Proposition 2.5, where we take $x_n = a_n$, $\hat{r}_n = \frac{1}{4}d_n$, and using (4.33) and (4.38), it is not difficult to see that (a_n, r_n) is a blow-up for $(u_n)_n$, and by using (4.27) and (4.33) we see that the $(k+1)$ blow-ups $(x_n^1, r_n^1), \dots, (x_n^k, r_n^k), (a_n, r_n)$ satisfy the property (\mathcal{P}) . This contradicts the fact that $k = \max A$. The proof of Proposition 4.4 is then complete. \square

Now we are in position to prove Theorem 1.1.

Proof of Theorem 1.1. Let (u_n, f_n) as in Theorem 1.1. If the first alternative in Theorem 1.1 does not hold, then by Proposition 4.4 there are k blow-ups $(x_n^1, r_n^1)_n, \dots, (x_n^k, r_n^k)_n$ with $1 \leq k \leq \frac{k_0}{16\pi^2}$, and k sequences $(R_n^1)_n, \dots, (R_n^k)_n$ of positive numbers satisfying (4.21)-(4.24) in Proposition 4.4. If we apply Proposition 4.3 by taking $S_n = 2 \operatorname{diam}(M)$, we obtain

$$\lim_{n \rightarrow +\infty} \int_{M \setminus \bigcup_{i=1}^k B_{R_n^i}(x_n^i)} e^{4u_n} dV_0 = 0, \quad (4.39)$$

which implies since the balls $B_{R_n^i}(x_n^i)$ are pairwise disjoint,

$$\lim_{n \rightarrow +\infty} \int_M e^{4u_n} dV_0 = \lim_{n \rightarrow +\infty} \sum_{i=1}^k \int_{B_{R_n^i}(x_n^i)} e^{4u_n} dV_0. \quad (4.40)$$

Since by (4.24) we have

$$\lim_{n \rightarrow +\infty} \int_{B_{R_n^i}(x_n^i)} e^{4u_n} dV_0 = \frac{16\pi^2}{k_0} \quad \forall i = 1, \dots, k, \quad (4.41)$$

and since $\int_M e^{4u_n} dV_0 = 1$, then we get from (4.40) and (4.41) that

$$k = \frac{k_0}{16\pi^2}. \quad (4.42)$$

On the other hand, since M is compact, then by passing to a subsequence, there exist m distinct points $a_1, \dots, a_m \in M$ with $m \leq k$ such that for any $i = 1, \dots, k$, the sequence $(x_n^i)_n$ converges to a limit in $\{a_1, \dots, a_m\}$. For any $i = 1, \dots, m$, if we set

$$l_i = \#\{j \in \llbracket 1, k \rrbracket : \lim_{n \rightarrow \infty} x_n^j = a_i\}, \quad (4.43)$$

then we have

$$l_1 + \dots + l_m = k = \frac{k_0}{16\pi^2}, \quad (4.44)$$

where we have used (4.42).

Let now $\varphi \in C^0(M)$. Then we have by (4.39)

$$\lim_{n \rightarrow +\infty} \int_M \varphi e^{4u_n} dV_0 = \lim_{n \rightarrow +\infty} \sum_{i=1}^k \int_{B_{R_n^i}(x_n^i)} \varphi e^{4u_n} dV_0. \quad (4.45)$$

But we have by the mean-value Theorem

$$\int_{B_{R_n^i}(x_n^i)} \varphi e^{4u_n} dV_0 = \varphi(y_n^i) \int_{B_{R_n^i}(x_n^i)} e^{4u_n} dV_0 \quad (4.46)$$

for some $y_n^i \in B_{R_n^i}(x_n^i)$. Since $R_n^i \rightarrow 0$ as $n \rightarrow \infty$, then we have

$$\lim_{n \rightarrow +\infty} y_n^i = \lim_{n \rightarrow +\infty} x_n^i \in \{a_1, \dots, a_m\}. \quad (4.47)$$

It follows from (4.45) by using (4.43), (4.44), (4.46) and (4.47) that

$$\lim_{n \rightarrow +\infty} \int_M \varphi e^{4u_n} dV_0 = \frac{16\pi^2}{k_0} \sum_{i=1}^m l_i \varphi(a_i).$$

This achieves the proof of Theorem 1.1. □

5. THE FLOW

In this section we prove our results concerning the Q -curvature flow. Through this section we assume that the total Q -curvature k_0 satisfies $k_0 > 0$ since $k_0 \leq 0$ is included in the case $k_0 \leq 16\pi^2$ which has been already proved by S. Brendle [3].

Lemma 5.1. *Let $u \in C^\infty(M \times [0, T])$ be the solution of problem (1.15) defined on a maximal interval $[0, T)$, and set*

$$A_t := \{x \in M : u(t, x) \geq \alpha_0\}, \quad t \in [0, T),$$

where $\alpha_0 = \frac{1}{4} \log \left(\frac{1}{2|M|} \int_M e^{4u_0} dV_0 \right)$, and where $|M|$ is the volume of (M, g_0) . For any $L_0 > 0$, there exists a positive constant C_0 depending only on L_0 and M such that, for any $T_0 \in [0, T)$, if

$$\|u_0\|_{H^2(M)} \leq L_0 \quad \text{and} \quad \inf_{t \in [0, T_0]} E(u(t)) \geq -L_0, \quad (5.1)$$

then A_t has volume $|A_t|$ (with respect to g_0) satisfying

$$|A_t| \geq \exp(-C_0 e^{2k_0 T_0}) \quad \text{for all } t \in [0, T_0]. \quad (5.2)$$

Proof. Through the proof of Lemma 5.1, C will denote a positive constant depending only on L_0 and M , whose value may change from line to line.

Since by (1.16) the volume of the conformal metric $e^{2u(t)}g_0$ remains constant, we may assume without loss of generality that for all $t \in [0, T]$

$$\int_M e^{4u(t)} dV_0 = 1. \quad (5.3)$$

Thus the first equation in (1.15) becomes

$$e^{4u} \partial_t u = -\frac{1}{2} (P_0 u + Q_0) + \frac{1}{2} k_0 e^{4u}. \quad (5.4)$$

Multiplying equation (5.4) by $u(t)$ and integrating on M with respect to dV_0 , and using (5.3), one gets

$$\frac{d}{dt} \int_M u e^{4u} dV_0 = -2 \int_M P_0 u \cdot u dV_0 - 2 \int_M Q_0 u dV_0 + 2k_0 \int_M u e^{4u} dV_0. \quad (5.5)$$

Let $L_0 \in \mathbb{R}$ and $T_0 \in [0, T]$ such that (5.1) is satisfied. Then we have for any $t \in [0, T_0]$

$$\frac{1}{2} \int_M P_0 u \cdot u dV_0 + \int_M Q_0 u dV_0 = E(u(t)) \geq -L_0. \quad (5.6)$$

It follows from (5.5) and (5.6) that

$$\frac{d}{dt} \int_M u e^{4u} dV_0 \leq - \int_M P_0 u \cdot u dV_0 + 2k_0 \int_M u e^{4u} dV_0 + 2L_0$$

which implies since P_0 is positive

$$\frac{d}{dt} \int_M u e^{4u} dV_0 \leq 2k_0 \int_M u e^{4u} dV_0 + 2L_0. \quad (5.7)$$

By setting $Y(t) = \int_M u e^{4u} dV_0$, it follows from (5.7) that for all $t \in [0, T_0]$,

$$Y(t) \leq \left(Y(0) + \frac{L_0}{k_0} \right) e^{2k_0 t} \leq C e^{2k_0 T_0}, \quad (5.8)$$

where the constant C depends only L_0 and M since $Y(0)$ depends only on the H^2 -norm of u_0 by Adams inequality (see section 2).

Since $u e^{4u} \geq -\frac{e^{-1}}{4}$, then we get from (5.8), for any $A \subset M$,

$$\int_A u e^{4u} dV_0 \leq C e^{2k_0 T_0}. \quad (5.9)$$

For $z > 0$, let $\varphi(z) = z \log z$. Then φ is convex on $(0, +\infty)$, and it satisfies for each $\lambda > 1$ and $z > 0$,

$$z = \frac{\varphi(\lambda z)}{\varphi(\lambda)} - \frac{\varphi(z)}{\log \lambda},$$

which implies, since $\varphi(z) \geq -e^{-1}$ for any $z > 0$,

$$z \leq \frac{\varphi(\lambda z)}{\varphi(\lambda)} + \frac{e^{-1}}{\log \lambda}. \quad (5.10)$$

For $t \in [0, T_0]$, let $A_t \subset M$ defined by

$$A_t = \{ x \in M : u(x, t) \geq \alpha_0 \}$$

where

$$\alpha_0 = \frac{1}{4} \log \left(\frac{1}{2|M|} \int_M e^{4u_0} dV_0 \right) = \frac{1}{4} \log \left(\frac{1}{2|M|} \right).$$

Since φ is convex, then it follows from Jensen inequality

$$\varphi \left(\frac{1}{|A_t|} \int_{A_t} e^{4u} dV_0 \right) \leq \frac{1}{|A_t|} \int_{A_t} \varphi(e^{4u}) dV_0 \quad (5.11)$$

But by (5.9) we have

$$\frac{1}{|A_t|} \int_{A_t} \varphi(e^{4u}) dV_0 \leq \frac{C e^{2k_0 T_0}}{|A_t|},$$

hence it follows from (5.11) that

$$\varphi \left(\frac{1}{|A_t|} \int_{A_t} e^{4u} dV_0 \right) \leq \frac{C e^{2k_0 T_0}}{|A_t|}. \quad (5.12)$$

Now, if $|A_t| \geq 1$, then the estimate (5.2) is trivially satisfied by taking C_0 any positive constant, and Lemma 5.1 is proved in this case. Thus we may suppose that $|A_t| < 1$. Then by using (5.10)

with $\lambda = \frac{1}{|A_t|}$ and $z = \int_{A_t} e^{4u} dV_0$, we have

$$\int_{A_t} e^{4u} dV_0 \leq \frac{|A_t|}{\log \frac{1}{|A_t|}} \varphi \left(\frac{1}{|A_t|} \int_{A_t} e^{4u} dV_0 \right) + \frac{e^{-1}}{\log \frac{1}{|A_t|}},$$

which gives by using (5.12)

$$\int_{A_t} e^{4u} dV_0 \leq (C e^{2k_0 T_0} + e^{-1}) \frac{1}{\log \frac{1}{|A_t|}} \leq \frac{C e^{2k_0 T_0}}{\log \frac{1}{|A_t|}}. \quad (5.13)$$

On the other hand, we have

$$1 = \int_M e^{4u} dv_0 = \int_{A_t} e^{4u} dV_0 + \int_{M \setminus A_t} e^{4u} dV_0 \quad (5.14)$$

and since $e^{4u} < e^{4\alpha_0} = \frac{1}{2|M|}$ on $M \setminus A_t$, then (5.14) implies

$$\frac{1}{2} \leq \int_{A_t} e^{4u} dV_0$$

which together with (5.13) give

$$\log \frac{1}{|A_t|} \leq C e^{2k_0 T_0}.$$

This achieves the proof Lemma 5.1. \square

Lemma 5.1 allows us to prove the following estimates on the solution :

Proposition 5.1. *Let $u \in C^\infty(M \times [0, T])$ be the solution of problem (1.15) defined on a maximal interval $[0, T)$. For any $L_0 > 0$, there exists a positive constant C_0 depending on L_0 and M such that, for any $T_0 \in [0, T)$, if*

$$\|u_0\|_{H^2(M)} \leq L_0 \quad \text{and} \quad \inf_{t \in [0, T_0]} E(u(t)) \geq -L_0,$$

then we have

$$\sup_{t \in [0, T_0]} \|u(t)\|_{H^2(M)} \leq \exp(C_0 e^{2k_0 T_0}). \quad (5.15)$$

Moreover, for any $k \in \mathbb{N}$, there exist a positive constant C_k depending on k, L_0, T_0 and M such that

$$\sup_{t \in [0, T_0]} \|u(t)\|_{H^k(M)} \leq C_k. \quad (5.16)$$

Proof. Through the proof of Proposition 5.1, C will denote a positive constant depending only on L_0 and M , whose value may change from line to line. For any measurable set $A \subset M$, we shall denote its volume with respect to the metric g_0 by $|A|$.

Since by (1.16) the volume of the conformal metric $e^{2u(t)}g_0$ remains constant, we may assume without loss of generality that

$$\int_M e^{4u(t)} dV_0 = 1. \quad (5.17)$$

This implies by using the elementary inequality $z \leq e^z$, that

$$\int_A u(t) dV_0 \leq \frac{1}{4} \int_A e^{4u(t)} dV_0 \leq \frac{1}{4} \quad (5.18)$$

for any $A \subset M$. Let $T_0 \in [0, T)$ and $L_0 > 0$ such that

$$\inf_{t \in [0, T_0]} E(u(t)) \geq -L_0.$$

If we let $A = A_t$, where A_t is as in Lemma 5.1, then we have by using (5.18) and the definition of the set A_t , for any $t \in [0, T_0]$,

$$\begin{aligned} \left| \int_M u(t) dV_0 \right| &\leq \left| \int_{A_t} u(t) dV_0 \right| + \left| \int_{M \setminus A_t} u(t) dV_0 \right| \\ &\leq C + \left| \int_{M \setminus A_t} u(t) dV_0 \right|. \end{aligned} \quad (5.19)$$

But by the Cauchy-Schwarz inequality we have

$$\left| \int_{M \setminus A_t} u(t) dV_0 \right| \leq |M \setminus A_t|^{\frac{1}{2}} \|u\|_{L^2(M)},$$

and by replacing this inequality in (5.19), we get for any $\varepsilon > 0$,

$$\left(\int_M u(t) dV_0 \right)^2 \leq (1 + \varepsilon) |M \setminus A_t| \|u(t)\|_{L^2(M)}^2 + C\varepsilon^{-1} + C. \quad (5.20)$$

Now, from Poincaré's inequality we have

$$\|u(t)\|_{L^2(M)}^2 \leq \frac{1}{\lambda_1} \int_M P_0 u(t) \cdot u(t) dV_0 + |M| |\bar{u}(t)|^2, \quad (5.21)$$

where λ_1 is the first positive eigenvalue of P_0 , and $\bar{u}(t) = \frac{1}{|M|} \int_M u(t) dV_0$ is the average of $u(t)$.

Thus it follows from (5.20) and (5.21) that

$$\left(1 - \frac{(1+\varepsilon)|M \setminus A_t|}{|M|}\right) \|u(t)\|_{L^2(M)}^2 \leq \frac{1}{\lambda_1} \int_M P_0 u \cdot u \, dV_0 + \frac{C}{|M|} \varepsilon^{-1} + \frac{C}{|M|}$$

that is

$$(|A_t| - \varepsilon|M \setminus A_t|) \|u(t)\|_{L^2(M)}^2 \leq \frac{|M|}{\lambda_1} \int_M P_0 u \cdot u \, dV_0 + C\varepsilon^{-1} + C. \quad (5.22)$$

Since by Lemma 5.1 we have $|A_t| \geq \exp(-C_0 e^{2k_0 T_0})$, then by choosing $\varepsilon = \frac{1}{2|M|} \exp(-C_0 e^{2k_0 T_0})$ in (5.22) and observing that $|M \setminus A_t| \leq |M|$, we obtain

$$\|u(t)\|_{L^2(M)}^2 \leq C \left(\int_M P_0 u(t) \cdot u(t) \, dV_0 + 1 \right) \exp(C_0 e^{2k_0 T_0}). \quad (5.23)$$

Since the functional E is decreasing along the flow by (1.17), then

$$\frac{1}{2} \int_M P_0 u(t) \cdot u(t) \, dV_0 + \int_M Q_0 u(t) \, dV_0 = E(u(t)) \leq E(u_0),$$

hence

$$\int_M P_0 u(t) \cdot u(t) \, dV_0 \leq C \|u(t)\|_{L^2(M)} + C. \quad (5.24)$$

It follows from (5.23) and (5.24) that

$$\|u\|_{L^2(M)}^2 \leq (\|u(t)\|_{L^2(M)} + 1) \exp(C e^{2k_0 T_0}),$$

which implies that

$$\|u(t)\|_{L^2(M)} \leq \exp(C e^{2k_0 T_0}). \quad (5.25)$$

Combining (5.24) and (5.25) we get (5.15). The higher order estimate (5.16) follows as in S. Brendle [3]. □

Proof of Theorem 1.2. Step 1 Global existence of the flow. Let $u \in C^\infty(M \times [0, T))$ be the solution of problem (1.15) defined on a maximal interval $[0, T)$, satisfying (1.18), that is

$$L := \inf_{t \in [0, T)} E(u(t)) > -\infty. \quad (5.26)$$

Suppose by contradiction that $T < +\infty$, then it follows from Proposition 5.1 by taking $L_0 = \|u_0\|_{H^2(M)} + |L|$ that

$$\sup_{t \in [0, T)} \|u(t)\|_{H^2(M)} < \exp(C_0 e^{2k_0 T}),$$

and for any $k \geq 2$:

$$\sup_{t \in [0, T)} \|u(t)\|_{H^k(M)} < +\infty. \quad (5.27)$$

It is clear that (5.27) implies that the solution $u(t)$ would be extended beyond T giving thus a contradiction. This proves Step 1.

Step 2 Convergence of the flow. According to the first step, the solution u is defined on $[0, +\infty)$, and (5.26) becomes

$$L := \inf_{t \in [0, +\infty)} E(u(t)) > -\infty. \quad (5.28)$$

Since by (1.16) the volume of the conformal metric $e^{2u(t)}g_0$ remains constant, we may assume without loss of generality that

$$\int_M e^{4u(t)} dV_0 = 1. \quad (5.29)$$

By using (1.17), we get for any $T > 0$,

$$\int_0^T \int_M e^{4u(t)} |\partial_t u(t)|^2 dV_0 dt = E(u_0) - E(u(T)) \leq E(u_0) - L,$$

which implies

$$\int_0^{+\infty} \int_M e^{4u(t)} |\partial_t u(t)|^2 dV_0 dt \leq E(u_0) - L. \quad (5.30)$$

By using the mean value theorem, we obtain from (5.30), that for any $n \in \mathbb{N}$, there exists $t_n \in [n, n+1]$ such that

$$\lim_{n \rightarrow +\infty} \int_M e^{4u(t_n)} |\partial_t u(t_n)|^2 dV_0 = 0. \quad (5.31)$$

Now if we set

$$u_n = u(t_n) \quad \text{and} \quad f_n = 2e^{4u_n} \partial_t u(t_n) + Q_0$$

then we have from (1.15)

$$P_0 u_n + f_n = k_0 e^{4u_n} \quad (5.32)$$

with

$$\int_M e^{4u_n} dV_0 = 1$$

and

$$\begin{aligned} \|f_n - Q_0\|_{L^1(M)} &\leq 2 \left(\int_M e^{4u_n} dV_0 \right)^{1/2} \left(\int_M e^{4u_n} |\partial_t u(t_n)|^2 dV_0 \right)^{1/2} \\ &= 2 \left(\int_M e^{4u_n} |\partial_t u(t_n)|^2 dV_0 \right)^{1/2} \xrightarrow{n \rightarrow +\infty} 0. \end{aligned}$$

Since we are supposing $k_0 \notin 16\pi^2 \mathbb{N}^*$, then we can apply Corollary 1.1 to get, for any $p \geq 1$

$$\int_M e^{p|u_n|} dV_0 \leq C_p, \quad (5.33)$$

which implies by using (5.31) that for any $q \in [1, 2)$

$$\|f_n\|_{L^q(M)} \leq C_q \quad (5.34)$$

for some constant C_q depending on q . Thus it follows from the elliptic regularity theory applied to equation (5.32) by using (5.33) and (5.34) that $(u_n)_n$ is bounded in $W^{4,q}(M)$ for any $q \in [1, 2)$. But by Sobolev embedding theorem we have $W^{4,q}(M) \subset C^\alpha(M)$ for any $\alpha \in (0, 1)$, and by applying the elliptic regularity theory again to equation (5.32), we obtain that $(u_n)_n$ is bounded in $H^4(M)$. In particular we have that $(u_n)_n$ is bounded in $H^2(M)$, that is

$$\|u_n\|_{H^2(M)} \leq C, \quad (5.35)$$

where C is a positive constant depending only on L, u_0 and M . Now, let us define $v_n(t) := u(t + t_n)$. Then v_n is a solution of problem (1.15) where u_0 is replaced by u_n , that is

$$\begin{cases} \partial_t v_n = -\frac{1}{2}e^{-4v_n} (P_0 v_n + Q_0) + \frac{k_0}{2} \\ v_n(0) = u_n. \end{cases} \quad (5.36)$$

We want to apply Proposition 5.1 to v_n . We have

$$\inf_{t \in [0,1]} E(v_n(t)) = \inf_{t \in [0,1]} E(u(t + t_n)) = \inf_{t \in [t_n, t_n+1]} E(u(t)) \geq L,$$

where L is given by (5.28). Then by Proposition 5.1, where we choose $T_0 = 1$ and $L_0 = |L| + C$ with C as in (5.35), there exist a positive constant C_0 depending on L, u_0 and M , such that

$$\sup_{t \in [0,1]} \|v_n(t)\|_{H^2(M)} \leq \exp(C_0 e^{2k_0}),$$

that is

$$\sup_{t \in [t_n, t_n+1]} \|u(t)\|_{H^2(M)} = \sup_{t \in [0,1]} \|v_n(t)\|_{H^2(M)} \leq \exp(C_0 e^{2k_0}),$$

and since $n \leq t_n \leq n+1$ for all $n \in \mathbb{N}$, then we have

$$\sup_{t \in [0, +\infty)} \|u(t)\|_{H^2(M)} \leq \exp(C_0 e^{2k_0}). \quad (5.37)$$

Following the argument of S. Brendle[3], one gets from (5.37) that,

$$\sup_{t \in [0, +\infty)} \|u(t)\|_{H^k(M)} \leq C_k$$

for any $k \geq 2$, and the convergence of the flow follows as in S. Brendle [3]. \square

Proof of Theorem 1.3. We proceed by contradiction. For $u \in C^\infty(M)$, let $\Phi(t, u)$ be the solution of (1.15) such that $\Phi(0, u) = u$, that is,

$$\begin{cases} \partial_t \Phi = -\frac{1}{2}e^{-4\Phi} (P_0 \Phi + Q_0) + \frac{1}{2} \frac{k_0}{\int_M e^{4\Phi} dV_0} \\ \Phi(0, u) = u. \end{cases} \quad (5.38)$$

Let $[0, T_u)$ be the maximal existence interval of Φ and suppose by contradiction that

$$\inf_{t \in [0, T_u)} E(\Phi(t, u)) = -\infty \quad \forall u \in C^\infty(M). \quad (5.39)$$

Let $X := C^\infty(M)$ endowed with its natural C^∞ topology, and let us introduce the sub-level set

$$X_0 := \{ u \in X : E(u) \leq -L \}, \quad (5.40)$$

where $L > 0$ is large enough. One fundamental property of X_0 is its invariance under the flow Φ , that is, if $u \in X_0$, then $\Phi(t, u) \in X_0$ for all $t \in [0, T_u)$, as it can be immediately checked by using the fact that E is decreasing along the flow Φ (see formula 1.17)

Following Z. Djadli and A. Machioldi [8], one can prove that X_0 is not contractible. Indeed, in [8] the set X_0 consists of H^2 - functions u satisfying $E(u) \leq -L$, but by following the same proof

as in [8], one can easily see that the same arguments work when considering C^∞ functions and the C^∞ topology on X_0 . Then we shall use our flow Φ to construct a deformation retraction from X onto X_0 , which would give a contradiction since X is contractible as a topological vector space.

By using (5.39) we can define for any $u \in X$

$$t_u = \min\{ t \in [0, T_u) : E(\Phi(t, u)) \leq -L \}. \quad (5.41)$$

Thus we have by using the continuity of Φ that

$$E(\Phi(t_u, u)) = -L. \quad (5.42)$$

We extend Φ on $[0, +\infty)$ by considering $\widehat{\Phi} : [0, +\infty) \times X \rightarrow X$ as follows

$$\widehat{\Phi}(t, u) = \begin{cases} \Phi(t, u) & \text{if } t \in [0, t_u] \\ \Phi(t_u, u) & \text{if } t \geq t_u. \end{cases}$$

By using Proposition 5.1 and the fact that the functional E is decreasing along the flow Φ (see formula (1.17)), one can prove that $\widehat{\Phi}$ is continuous on $[0, +\infty) \times X$.

We define now the following homotopy map : $H : [0, 1] \times X \rightarrow X$ by

$$H(t, u) = \begin{cases} \widehat{\Phi}(\frac{t}{1-t}, u) & \text{if } t \in [0, 1) \\ \widehat{\Phi}(t_u, u) & \text{if } t = 1. \end{cases}$$

Then it is easy to see that we have

$$\begin{cases} H(0, u) = u \quad \forall u \in X, \\ H(t, u) = u \quad \forall u \in X_0, \quad \forall t \in [0, 1] \\ H(1, u) \in X_0 \quad \forall u \in X. \end{cases}$$

This proves that X_0 is a deformation retract of X which is impossible since X_0 is non contractible. The proof of Theorem 1.3 is then complete. \square

Proof of Theorem 1.4. Let \mathcal{S} be the set of all solutions of the Q -curvature equation

$$P_0 u + Q_0 = k_0 e^{4u} \quad (5.43)$$

such that

$$\int_M e^{4u} dV_0 = 1$$

(we note here that by using (5.43), the last condition is automatically satisfied when $k_0 \neq 0$).

According to Corollary 1.1, we have for any $k \in \mathbb{N}$

$$\|u\|_{C^k(M)} \leq C_k \quad \forall u \in \mathcal{S} \quad (5.44)$$

where C_k is a positive constant independent of u . It follows from (5.44) that the functional E satisfies

$$E(u) \geq L \quad \forall u \in \mathcal{S} \quad (5.45)$$

for some constant $L \in \mathbb{R}$ independent of u .

Let $\lambda < L$, we shall prove that for any $u_0 \in C^\infty(M)$ with $E(u_0) \leq \lambda$, the solution $u(t)$ of (1.15) such that $u(0) = u_0$, satisfies

$$\lim_{t \rightarrow T} E(u(t)) = -\infty, \quad (5.46)$$

where $[0, T)$ is the maximal existence interval of u . Indeed, suppose by contradiction that (5.46) does not hold. Then according to Theorem 1.2, we have $T = +\infty$, and the solution $u(t)$ converges (as $t \rightarrow +\infty$) to a function $u_\infty \in C^\infty(M)$ satisfying

$$P_0 u_\infty + Q_0 = \frac{k_0}{\int_M e^{4u_\infty} dV_0} e^{4u_\infty}. \quad (5.47)$$

Moreover, since the functional E is decreasing along the flow, u_∞ satisfies

$$E(u_\infty) \leq E(u_0) \leq \lambda. \quad (5.48)$$

On the other hand, since E is translation invariant, that is, $E(u + c) = E(u) \quad \forall c \in \mathbb{R}$, we may assume by adding an appropriate constant to u_∞ , that $\int_M e^{4u_\infty} dV_0 = 1$. This implies by using (5.47) that $u_\infty \in \mathcal{S}$. Thus it follows from (5.45) that

$$E(u_\infty) \geq L$$

which contradicts (5.48) since $\lambda < L$. This achieves the proof of Theorem 1.4. □

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